Symplectic reflection algebras and Poisson geometry

by

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For Mum, Dad, and Rita.

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Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow.

Chapters 1 and 2 and Section 3.3 cover notation, definitions and known results.

Chapters 3, 4 and 5 are the author's original work with the exception of those results which are explicitly referenced. Some of the results therein have been published in [57] and will also appear in [56].

Summary

The subject of this thesis is the Poisson geometry of varieties associated to the centres of symplectic reflection algebras in the PI case. In particular it focuses on describing the symplectic leaves of these varieties.

Chapter 1 introduces the theory of symplectic reflection algebras. We introduce the classical objects of rings of invariants and skew group rings and describe deformations of these and present some of their basic properties following [26]. We highlight a dichotomy in the theory and focus our attention to the PI case. The framework for studying the representation theory is developed and the close connection with Poisson geometry is explained using results of [11]. Poisson algebras are introduced and the fundamental notion of stratifying Poisson varieties by symplectic leaves is explained. Although symplectic leaves are a well-known concept in the field of Poisson manifolds (see [55] and [79]) we examine them in the context of Poisson algebraic geometry as described in [11].

At the heart of Chapter 2 is the example of representations of deformed preprojective algebras and we view different aspects of the geometry of these varieties. The necessary invariant theory is introduced, which involves discussing the categorical quotient and its properties as in [68] or [48]. Crucial to us will be the stratification by orbit type. Moment maps and Marsden-Weinstein reductions for symplectic varieties are introduced following the approach of [15]. Combinatorial aspects of quivers are examined, in particular this includes the description by root vectors of orbit type strata for representations of preprojective algebras, as given in [16] and [17]. Finally hyper-Kähler manifolds make an appearance and provide a means for establishing the existence of the local normal form of the moment map, as was proved in [63]. The local normal form plays a crucial role in our main theorem concerning symplectic leaves, Theorem 4.2.

Chapter 3 combines filtered and graded techniques with the properties of symplectic leaves to

establish that the associated variety of a Poisson prime ideal of the centre of a symplectic reflection algebra is irreducible. The arguments used are based on proofs of similar results for enveloping algebras of finite dimensional complex semisimple Lie algebras, [78], and for the non PI case of symplectic reflection algebras, [29]. We show how, in principal, this allows us to describe the symplectic leaf corresponding to a Poisson prime ideal in terms of a certain conjugacy class of the group. This chapter has been published in [57].

Chapter 4 begins with an explicit description (modelled on work of [74] for real symplectic manifolds) of the symplectic leaves for certain types of Marsden-Weinstein reductions, which includes representation spaces of deformed preprojective algebras. In this latter case these are the orbit type strata so the leaves can be described in terms of root vectors for the quiver. We apply this to symplectic reflection algebras by proving an isomorphism, which extends a result of [26], between these representation spaces and the varieties associated to centres of certain symplectic reflection algebras, namely those corresponding to wreath products. This isomorphism identifies symplectic leaves and so enables us to understand the symplectic leaves in terms of roots of a quiver. This chapter will appear in [56].

Chapter 5 contains some examples based on the results of Chapter 4. We calculate, in the case of the wreath product of the symmetric group with a cyclic group of order two, the values of deformation parameter at which the varieties associated to centres of symplectic reflection algebras are singular. We also calculate the number and dimensions of symplectic leaves. We compare this information to representations of the symplectic reflection algebra. We finish by posing some questions concerning the representation theory lying above a symplectic leaf.

Introduction

The aim of this thesis is to investigate certain affine algebraic Poisson varieties arising in the study of symplectic reflection algebras. Poisson varieties have many different facets: on one hand they are defined in a purely algebraic way, but they define natural geometric constructions and so lead one into the algebraic and differential geometric viewpoints. We use each of these approaches in this thesis. The algebraic side leads us to filtered and associated graded techniques as well as quivers; the geometric side encompasses invariant theory and symplectic reduction. A fundamental structure which exists on all Poisson varieties is the stratification by symplectic leaves and most of our theorems are aimed at better understanding the leaves of the varieties we study.

Symplectic reflection algebras were introduced by Etingof and Ginzburg, [26]. Given a symplectic vector space, V and a finite subgroup $G \subset \mathrm{Sp}(V)$, the symplectic reflection algebras, $H_{t,c}$, are a family of deformations of the skew group algebra $\mathbb{C}[V] * G$. They were motivated by the representation theory, geometry and integrable systems which are related to these algebras.

When the parameter t equals zero then geometry comes to the fore. The centre, $Z_{0,c}$, of $H_{0,c}$ is an affine Poisson algebra and $H_{0,c}$ is finitely generated over its centre. Then any irreducible $H_{0,c}$-module is finite dimensional and so its annihilator in $Z_{0,c}$ is a maximal ideal by Schur's Lemma. Let $\mathrm{irr}\ H_{0,c}$ denote the set of isomorphism classes of irreducible $H_{0,c}$-modules. Let $X_{0,c} = \mathrm{Maxspec}\ Z_{0,c}$, we have

$$\mathrm{irr}\ H_{0,c} = \bigcup_{\mathfrak{m} \in X_{0,c}} \mathrm{irr}\ (H_{0,c}/\mathfrak{m}H_{0,c}). \tag{0.1}$$

We can consider one maximal ideal at a time and examine the irreducible modules of the finite dimensional algebra $H_{0,c}/\mathfrak{m}H_{0,c}$. It is here that the Poisson geometry comes into play. The variety $X_{0,c}$ is Poisson and so can be stratified into its symplectic leaves. It was proved by Brown and

Gordon, [11], that if $\mathfrak{m}, \mathfrak{n}$ lie in the same symplectic leaf then there is an algebra isomorphism

$$H_{0,c}/\mathfrak{m}H_{0,c} \cong H_{0,c}/\mathfrak{n}H_{0,c}. \tag{0.2}$$

There are two questions which are now important.

A. How can we describe the symplectic leaves of $X_{0,c}$?

B. What can we say about the sets irr $(H_{0,c}/\mathfrak{m}H_{0,c})$ or the isomorphism classes of the algebras $H_{0,c}/\mathfrak{m}H_{0,c}$ corresponding to each leaf?

As the theory of symplectic reflection algebras is still in its infancy there has not been much progress made on either of these questions. In general it is known that $X_{0,c}$ has finitely many symplectic leaves and that this implies that the leaves are locally closed in the Zariski topology, [11]. In particular the smooth locus is a leaf and the corresponding factor algebras are isomorphic to matrix algebras over $\mathbb{C}$ of size $|G| \times |G|$, [26] and [9]. In the undeformed case, $H_{0,0} = \mathbb{C}[V] * G$ and $X_{0,0} = V/G$, the symplectic leaves have been calculated and the structure of the factors $H_{0,0}/\mathfrak{m}H_{0,0}$ described, [11]. The symplectic leaves can be given explicitly. Let $\pi : V \to V/G$ be the orbit map. Then the symplectic leaves of V/G are the sets

$$(V/G)_{(H)} := \{Gv \in V/G : G_v \text{ is conjugate to } H\} \tag{0.3}$$

where $H \leq G$. Thus each leaf in V/G is labelled by the conjugacy class of a subgroup of G.

In this thesis we will focus mainly on aspects of question **A**, but question **B** provides much of the underlying motivation. We make two contributions towards question **A**. Our first stems from the relationship between $Z_{0,c}$ and $Z_{0,0}$. The algebra $H_{0,c}$ has a natural filtration so that $\operatorname{gr} H_{0,c} \cong \mathbb{C}[V] * G$, and if we restrict this filtration to the subalgebra $Z_{0,c}$ we find that $\operatorname{gr} Z_{0,c} \cong \mathbb{C}[V]^G \subset \mathbb{C}[V] * G$. We prove that, for any V and G, if P is an ideal of $Z_{0,c}$ which is prime and stable under the Poisson bracket then $\sqrt{\operatorname{gr} P}$ is also prime and Poisson, Corollary 3.11. So we get a map

$$\begin{gathered} \{\text{Poisson prime ideals of } Z_{0,c}\} \to \{\text{Poisson prime ideals of } \mathbb{C}[V]^G\} \\ P \mapsto \sqrt{\operatorname{gr} P}. \end{gathered} \tag{0.4}$$

The defining ideal of the closure of symplectic leaf in $X_{0,c}$ is Poisson and prime. On the other hand the smooth locus of the variety defined by a Poisson prime ideal is a symplectic leaf. This allows us to define a map

$$\begin{gathered}\{\text{symplectic leaves of } X_{0,c}\} \to \{\text{symplectic leaves of } V/G\} \\ \mathcal{S} = \text{sm } \mathcal{V}(P) \mapsto P \mapsto \sqrt{\operatorname{gr} P} \mapsto \text{sm } \mathcal{V}(\sqrt{\operatorname{gr} P}).\end{gathered} \tag{0.5}$$

Therefore by (0.3) and (0.5) we have a map

$$\Omega : \{\text{symplectic leaves of } X_{0,c}\} \to \{\text{conjugacy classes of subgroups of } G\}.$$

This result has a direct analogue in Lie theory. Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. Then the associated variety of a primitive ideal of $\mathcal{U}(\mathfrak{g})$ is irreducible, [7] and [42]. Another example similar our to situation also comes from Lie theory. Let $\mathfrak{n}$ be a finite dimensional solvable Lie algebra over $\mathbb{C}$. Then $S\mathfrak{n}$ is a Poisson algebra with bracket induced by the Lie bracket. There is a bijection between the Poisson prime ideals of $S\mathfrak{n}$ and the set of primitive ideals of $\mathcal{U}(\mathfrak{n})$ which is bicontinuous for topologies of the two spaces, that is, the Zariski topology and the Jacobson topology respectively, see [23] for the proof when $\mathfrak{n}$ is nilpotent and [58] for the general case. This map is called the Dixmier map. Our version of such a map is (0.4). It is easy to find examples such that (0.4) is neither injective nor surjective, Remark 3.5 1. However, we do find a class of examples where (0.4) in injective, Corollary 5.8.

If we take $\mathfrak{n}$ to be nilpotent then we have a further comparison arising from the orbit method, see [45]. The Poisson algebra $S\mathfrak{n}$ is the coordinate ring of $\mathfrak{n}^*$, and so the latter is a Poisson variety. The symplectic leaves are the coadjoint orbits. In [22] Dixmier showed how to associate to each orbit a primitive ideal of $\mathcal{U}(\mathfrak{n})$. Thus the map (0.5) is an approximation to the orbit method, although this approximation is very coarse. Perhaps the best one could hope for is that knowledge about a conjugacy class of subgroups, $[H]$ say, gives some information concerning $\Omega^{-1}([H])$.

A slightly different point of view is the connection between $H_{0,c}$ and the geometry of $X_{0,c}$. There are two other examples of affine PI algebras whose centres are Poisson algebras. One is that of enveloping algebras of finite dimensional Lie algebras in characteristic $p > 0$. Although symplectic leaves belong to the world of differential geometry so are not defined in characteristic p, one can think of a coadjoint orbit as replacing the notion of symplectic leaf. There is an explicit relationship between the dimension of a coadjoint orbit and the dimension of a corresponding representation, [65]. On the other hand, quantum groups at roots of unity defined over $\mathbb{C}$ are usually affine PI algebras and their centres are Poisson. There are formulae relating dimensions of symplectic leaves of maximal dimension and the dimensions of corresponding irreducible representations in both the case of quantised enveloping algebras, [20], and quantised function algebras, [21]. We do not expect that the symplectic leaves of $X_{0,c}$ should necessarily give information about the dimensions of irreducible representations, but it would be interesting to know what the relationship is between leaves and corresponding modules in our situation.

To describe our second contribution to question **A**, we return to the problem of how to describe the leaves of $X_{0,c}$. Let Γ be a finite subgroup of $SL_2(\mathbb{C})$ acting on its natural representation. The vector space $\mathbb{C}^2$ can be endowed with a symplectic form and Γ preserves this. Symplectic reflection algebras for such pairs $(\mathbb{C}^2, \Gamma)$ were actually first studied (under a different name) by Crawley-Boevey and Holland in [19]. Their approach was to use the McKay graph of Γ to associate to $(\mathbb{C}^2, \Gamma)$ a family of noncommutative algebras called deformed preprojective algebras. These are Morita equivalent to $H_{0,c}$ and their representation spaces (at an appropriate dimension vector) are isomorphic to $X_{0,c}$. Thus the combinatorial structure of quivers can be brought to bear on the geometry of $X_{0,c}$.

One can consider a more general version of this situation, that is, the wreath product $G = S_n \wr \Gamma$ acting on $V = (\mathbb{C}^2)^{\oplus n}$. Modelled on results in [26] and [19], we prove in Chapter 4 that for (V, G) the variety $X_{0,c}$ is isomorphic to certain representations of deformed preprojective algebras, Corollary 4.25. Thus we can use the structure of quivers to describe $X_{0,c}$. In particular the symplectic leaves of $X_{0,c}$ are identified with representation type strata, which are described in terms of roots of the quiver. These latter strata have been studied in [16] and we show by the example in Chapter 5 how can one use these results to calculate basic numerical information about the leaves of $X_{0,c}$, Proposition 5.7.

The thesis is organised as follows. The definition of symplectic reflection algebras and relevant notions from Poisson geometry are given in Chapter 1. Chapter 2 introduces deformed preprojective algebras and their representation spaces. The irreducibility of associated varieties is proved in Chapter 3. The isomorphism between deformed quotient singularities for wreath products and representations of deformed preprojective algebras is proved in Chapter 4. Chapter 5 contains calculations of dimensions of symplectic leaves for wreath products of cyclic groups and a comparison with some irreducible representations. We finish each chapter with bibliographic and supplementary remarks, Chapter 5 concludes with some questions relating to question **B**.

Contents

Chapter 1

Symplectic reflection algebras

We introduce the main motivation for our work: symplectic reflection algebras. This family of algebras was introduced in [26] and their properties vary according to choices of deformation parameter. We present some basic properties common to all symplectic reflection algebras and highlight the dichotomy which occurs at an early stage in this theory. We introduce Poisson geometry and investigate how it plays a leading role in the representation theory of certain symplectic reflection algebras.

1.1 Notation and conventions

A good reference for standard notions in noncommutative algebra is [60]. Our algebras are all $\mathbb{C}$-algebras, associative and unital, and we do not require that a subalgebra of an algebra shares the same identity element. We say that an algebra is *affine* if it is finitely generated as an algebra over $\mathbb{C}$. For an algebra, A, with subalgebra B, we will say that A is finite over B to mean that A is a finitely generated B-module. For a $\mathbb{C}$-algebra, A, we assume that a filtration, $\mathcal{F}$, of A is a $\mathbb{Z}$-filtration and that $\dim_{\mathbb{C}} \mathcal{F}_i < \infty$ for all $i \in \mathbb{Z}$. We will denote the associated graded algebra with respect to $\mathcal{F}$ by $\operatorname{gr}^{\mathcal{F}} A$, or simply by $\operatorname{gr} A$ if there is no risk of confusion. The A-module of $\mathbb{C}$-linear derivations $A \to A$ will be written $\operatorname{Der}_{\mathbb{C}} A$. For an algebra, A, we write $GK \operatorname{Dim} A$ for the Gelfand-Kirillov dimension of A: see [49] for the definition and basic properties.

We refer the reader to [38] for basic definitions in algebraic geometry. The varieties in this thesis are defined over $\mathbb{C}$ and are not necessarily irreducible. By an affine algebraic variety we mean the set of maximal ideals of an affine commutative algebra, R, which we denote by $\operatorname{Max} R$, whose coordinate ring is R/I, where I equals the radical of R. Occasionally we will work with the affine scheme $\operatorname{Spec} R$, given its structure of a reduced scheme, and we identify $\operatorname{Max} R$ and $\operatorname{Spec} R$. For an ideal, I, of R we denote by $\mathcal{V}(I)$ the closed subvariety of $\operatorname{Max} R$, $\operatorname{Max} R/I$. For an algebraic variety, X, we denote by $\operatorname{Dim} X$ its dimension, by $\operatorname{sm}X$ its smooth locus, and by $\operatorname{sing}X$ its singular locus, that is $\operatorname{sing}X = X \setminus \operatorname{sm}X$. For any subset Y of X we will denote the closure of Y in the Zariski topology by $\overline{Y}$. We denote by $\mathcal{O}_X$ the structure sheaf of X, and we write simply $\mathcal{O}$ when there is no risk of confusion. Let $U \subseteq V$ be open subsets of X. Then for $f \in \mathcal{O}_X(V)$ the restriction of f to U will be written $f|_U$. The *tangent space* of $x \in X$, T_xX, is the vector space $\operatorname{Der}_{\mathbb{C}}(\mathcal{O}(X)_x, \mathcal{O}(X)_x/m_x)$, where $\mathcal{O}(X)_x$ is the local ring of x with maximal ideal m_x. When X is smooth we denote by TX the tangent bundle of X, and for each $x \in X$, T_xX is the tangent space of x in X. An n-vector field is a morphism $X \to \bigwedge^n TX$ which is a section to the bundle map $\bigwedge^n TX \to X$. Similarly, let T^*X denote the cotangent bundle; then an n-form is a section $X \to \bigwedge^n T^*X$. We will sometimes write vector field to mean 1-vector field. For a smooth affine algebraic variety, X, there is an isomorphism of vector spaces between $\operatorname{Der}_{\mathbb{C}} \mathcal{O}(X)$ and the space of all vector fields, see [38, Exercise 5.18]. We can give this explicitly. Let $\delta \in \operatorname{Der}_{\mathbb{C}} \mathcal{O}(X)$ and $x \in M$. We write $\bar{\delta} \in \operatorname{Der}_{\mathbb{C}}(\mathcal{O}(X)_x, \mathcal{O}(X)_x)$ for the unique extension of δ to $\mathcal{O}(X)_x$, and let π be the quotient map $\mathcal{O}(X)_x \to \mathcal{O}(X)_x/m_x$; then the isomorphism is given by the map $\delta \mapsto (x \mapsto \pi \circ \bar{\delta})$. For a smooth variety, Y, which is not necessarily affine a vector field $\xi : Y \to TY$ defines a derivation of $\mathcal{O}(Y)$. Let $y \in Y$ and $f \in \mathcal{O}(Y)$. Let $\xi_y = \xi(y)$ and denote the image of f in $\mathcal{O}(Y)_y$ by f_y. Then $(\xi \cdot f)(y) = \xi_y(f_y)$.

By a manifold, we mean a complex manifold. Several good references exist for these, such as [2] or [27]. For a manifold, M, we denote the sheaf of holomorphic functions by $\mathcal{O}_M^{hol}$, or simply by $\mathcal{O}^{hol}$, and its dimension as a complex manifold by $\operatorname{Dim} M$. For a holomorphic map $f : M \to N$ between manifolds we denote the differential of f by df. The *tangent space* of x is written T_xM and is the vector space $\operatorname{Der}_{\mathbb{C}}(\mathcal{O}_x^{hol}, \mathcal{O}_x^{hol}/\mathbf{m}_x)$, where $\mathcal{O}_x^{hol}$ is the local algebra of germs of functions around x with maximal ideal, $\mathbf{m}_x$. One can think of $\mathcal{O}_x^{hol}$ as the algebra of power series (in Dim M variables) which converge in a neighbourhood of x and $\mathbf{m}_x$ as the power series with no constant term. For $x \in M, f \in \mathcal{O}^{hol}$ let $[f]$ be the power series expansion of f around x. The map $f \mapsto [f]$ defines a homomorphism of algebras $\mathcal{O}^{hol} \to \mathcal{O}_x^{hol}$.

We denote the tangent manifold of M by TM and the cotangent manifold by T^*M. As in the case of smooth varieties we have the notion of n-vector fields and n-forms which are holomorphic maps $M \to \bigwedge^n TM$ and $M \to \bigwedge^n T^*M$ respectively, which are sections for the corresponding bundle maps. We will write vector field to mean a 1-vector field.

For any vector field, ξ, there is an action of ξ on $\mathcal{O}^{hol}(M)$ such that ξ is a $\mathbb{C}$-linear derivation of $\mathcal{O}^{hol}(M)$: for any $x \in M$ and $f \in \mathcal{O}^{hol}(M)$ we put $(\xi \cdot f)(x) = \xi_x([f])$. This defines a vector space isomorphism between $\operatorname{Der}_{\mathbb{C}} \mathcal{O}^{hol}(M)$ and the space of vector fields on M, [1, Theorem 2.2.10]. An *integral curve* to ξ at $x \in M$ is a holomorphic map, ϕ, between $B_\epsilon := \{t \in \mathbb{C} : |t| < \epsilon\}$ for some ϵ and M such that $\phi(0) = x$ and $d\phi(t) = \xi(\phi(t))$ for all $t \in B_\epsilon$. Integral curves to ξ at x exist and are unique by the Picard Theorem for ODEs, [2, Theorem 2.1].

A smooth variety, X, can naturally be given the structure of a manifold. Then we have both the algebraic tangent bundle, $T^{alg}X$, and the holomorphic tangent bundle, $T^{hol}X$. To distinguish between n-vector fields and n-forms associated to $T^{alg}X$ and those associated to $T^{hol}X$, we shall call the former *algebraic* and the latter *holomorphic*. There is holomorphic map $T^{alg}X \to T^{hol}X$ which makes the following diagram commute

$$\begin{tikzcd}
T^{alg}X \arrow[r] \arrow[d] & T^{hol}X \arrow[dl] \\
X &
\end{tikzcd}$$

and therefore every algebraic vector field induces a holomorphic one. Similarly, algebraic n-vector fields and n-forms induce corresponding holomorphic ones.

1.2 Invariants and skew group algebras

Let V be a finite dimensional complex vector space, and let G be a finite subgroup of $\mathrm{GL}(V)$, the general linear group of V. Consider $\mathcal{O}(V)$, the coordinate ring of V, which is a polynomial algebra, and whose i^{th}-graded piece we denote by $\mathcal{O}(V)_i$. The action of G on V induces an action of G on $\mathcal{O}(V)$ by $f^g(v) = f(g^{-1}v)$ for all $g \in G, f \in \mathcal{O}(V)$ and $v \in V$. There are two algebras which arise naturally in this situation. Firstly we define the *ring of polynomial invariants*,

$$\mathcal{O}(V)^G := \{f \in \mathcal{O}(V) : f^g = f \text{ for all } g \in G\}. \tag{1.1}$$

By [5, Remark, Page 8] this is a subalgebra of $\mathcal{O}(V)$ which can be viewed as the coordinate ring of V/G, the space of orbits of G in V. There are a number of beautiful results concerning the ring of invariants, and we list some below. For any $g \in G$ let $\mathrm{Fix}_V\,(g)$ be the set $\{v \in V : g{\cdot}v = v\}$, which is a vector subspace of V. The set of *complex reflections* is $\{g \in G : \mathrm{Fix}_V(g)$ has codimension 1 in $V\}$.

Theorem 1.1.

(i) $\mathcal{O}(V)^G$ *is a graded domain.*

(ii) $\mathcal{O}(V)^G$ *is a finitely generated* $\mathbb{C}$*-algebra and* $\mathcal{O}(V)$ *is a finite* $\mathcal{O}(V)^G$*-module.*

(iii) $\mathcal{O}(V)^G$ *has finite global dimension if and only if* G *is generated by complex reflections.*

(iv) $\mathcal{O}(V)^G$ *has GK dimension equal to* $\mathrm{Dim}_{\mathbb{C}}\, V$.

Proof.

(i) $\mathcal{O}(V)$ is a domain so the subalgebra, $\mathcal{O}(V)^G$, is a domain also. Since G acts on V, the action of G on $\mathcal{O}(V)$ preserves the graded pieces of $\mathcal{O}(V)$, therefore $\mathcal{O}(V)^G$ is graded.

(ii) This is the Hilbert-Noether Theorem, see [5, Theorem 1.3.1].

(iii) $\mathcal{O}(V)^G$ has finite global dimension if and only if it is a polynomial ring in finitely many indeterminates by Hilbert's Syzygy Theorem, [5, Corollary 4.2.3], and [5, Theorem 6.2.2 (b)]. $\mathcal{O}(V)^G$ is an affine polynomial ring if and only if G is generated by complex reflections by the Chevelley-Shephard-Todd Theorem, [5, Theorem 7.2.1].

(iv) This follows from the fact that $\mathcal{O}(V)$ has GK dimension $\mathrm{Dim}_{\mathbb{C}}\, V$ and from part (ii), see [60, Proposition 8.2.9 (ii)].

$\square$

Rephrasing (iii) in geometric language, the variety V/G is smooth if and only if G is generated by complex reflections.

We now turn to a second algebra which arises in this context and is closely related to the ring of invariants. It arises as an example of a more general construction which we introduce now.

Definition 1.2. *Let A be a $\mathbb{C}$-algebra and let G be a finite group which acts faithfully on A by algebra automorphisms. For $a \in A, g \in G$ let $a^g = g(a)$. The skew group algebra, $A * G$, is the free left A-module with basis the elements of G, and multiplication defined by $(ag)(bh) = ab^g gh$ for $a, b \in A,\ g, h \in G$.*

Properties of the algebra A are frequently transferred to a skew group algebra of A.

Theorem 1.3. *The following properties, when possessed by A, are inherited by $A * G$:*

(i) being Noetherian;

(ii) having GK dimension r;

(iii) having global dimension r.

Furthermore, if the group of units of A is central in A then the following properties are inherited from A by $A * G$:

(iv) being simple;

(v) being prime.

Proof. (i) is [64, Proposition 1.6]; (ii) is [60, Proposition 8.2.9(iv)]; (iii) is [60, Theorem 7.5.6(iii)]; (iv) is [60, Proposition 7.8.12] and (v) is [64, Theorem 12.9]. $\square$

Returning to our setup, we can form the skew group algebra, $\mathcal{O}(V) * G$. It follows from the above theorem that this a prime Noetherian algebra whose GK dimension and global dimension are equal to $\mathrm{Dim}_{\mathbb{C}} V$. From the definition of the skew group ring it is clear that there is a vector space isomorphism, $\mathcal{O}(V) * G \cong \mathcal{O}(V) \otimes_{\mathbb{C}} \mathbb{C}G$, where $\mathbb{C}G$ is the group algebra of G. Therefore every element of $\mathcal{O}(V) * G$ can be written uniquely in the form $\sum_{g \in G} f_g g$ where $f_g \in \mathcal{O}(V), g \in G$, and we shall consider $\mathcal{O}(V)$ as a subalgebra of $\mathcal{O}(V) * G$ via the monomorphism $f \mapsto f 1_{\mathbb{C}G}$. $\mathcal{O}(V) * G$ is graded by defining the i^{th}-graded piece to be $\{\sum_{g \in G} f_g g : f_g \in \mathcal{O}(V)_i \text{ for all } g \in G\}$. We now express the relationship between the ring of invariants and the skew group algebra. Let $e = \frac{1}{|G|} \sum_{g \in G} g$ be the symmetrising idempotent of $\mathbb{C}G$.

Lemma 1.4. *Let $Z(\mathcal{O}(V)*G)$ be the centre of the skew group algebra. Then $Z(\mathcal{O}(V)*G) = \mathcal{O}(V)^G$ and $\mathcal{O}(V) * G$ is a finitely generated $\mathcal{O}(V)^G$-module. There is an algebra isomorphism*

$$\begin{aligned} \mathcal{O}(V)^G &\longrightarrow \quad e(\mathcal{O}(V) * G)e \\ z \quad &\longmapsto \qquad ze. \end{aligned}$$

Proof. It is straightforward to prove that $\mathcal{O}(V)^G \subseteq Z(\mathcal{O}(V) * G)$. On the other hand let $f = \sum_{g\in G} f_g g \in Z(\mathcal{O}(V) * G)$. Let $1 \neq h \in G$ and choose some $x \in V^*$ such that $x^h \neq x$. Now $xf = fx$ implies that $\sum_{g\in G} xf_g g = \sum_{g\in G} f_g gx = \sum_{g\in G} f_g x^g g$ and therefore that $f_h(x - x^h) = 0$. Since $x^h \neq x$ this means that $f_h = 0$. Thus for all $1 \neq h \in G$, $f_h = 0$. Therefore $f = f_1 1 \in \mathcal{O}(V)$. The fact that f commutes with all $g \in G$ means that $f^g = f$ for all $g \in G$ and so $f \in \mathcal{O}(V)^G$.

Since G is finite, $\mathcal{O}(V) * G$ is a finite $\mathcal{O}(V)$-module, and $\mathcal{O}(V)$ is a finite $\mathcal{O}(V)^G$-module by Theorem 1.1 (ii). Therefore $\mathcal{O}(V) * G$ is a finite module over $\mathcal{O}(V)^G$.

Since any $z \in Z(\mathcal{O}(V) * G)$ commutes with e, and e is an idempotent, the map $z \mapsto ze$ is an algebra homomorphism. It is injective since for all $f \in \mathcal{O}(V)$, $fe = 0$ implies that $f = 0$. It is surjective because for $\sum_{g\in G} f_g g \in \mathcal{O}(V)*G$, $e(\sum_{g\in G} f_g g)e = e(\sum_{g\in G} f_g)e$. Now for any $f \in \mathcal{O}(V)$, $efe = f'e$ where $f' = \frac{1}{|G|}\sum_{g\in G} f^g \in \mathcal{O}(V)^G = Z(\mathcal{O}(V) * G)$. □

We note that the identity element of $e(\mathcal{O}(V) * G)e$ is e and so it does not share the identity of $\mathcal{O}(V) * G$. It does inherit a grading from the skew group algebra though, because e lies in the degree zero part of the grading on $\mathcal{O}(V) * G$.

1.3 Definition and properties of symplectic reflection algebras

Having considered rings of invariants and skew group algebras, we are in a position to define a whole family of algebras which are deformations of these more classical objects. To do this we must restrict our attention to a more rigid hypothesis (although one can form appropriate deformations under slightly weaker conditions, see [66]). Suppose that V is a finite dimensional complex vector space, endowed with a complex symplectic form ω, and let G be a finite subgroup of $Sp(V)$, the symplectic group of V. It will sometimes be convenient to work with a symplectic basis of V, $\{x_1, \dots, x_n, y_1, \dots, y_n\}$, that is, the x_is and y_js form a basis of V and $\omega(x_i, x_j) = \omega(y_i, y_j) = 0$, $\omega(x_i, y_j) = \delta_{ij}$ for all i, j. The form, ω, allows us to construct an isomorphism of G-modules, $V \cong V^*$, which means that we can (and will) identify $\mathcal{O}(V)$ with SV, the symmetric algebra on V.

Lemma 1.5. *For any $g \in G$, $\mathrm{Fix}_V(g)$ is a symplectic subspace of V.*

Proof. One can define an alternating form ω' on $\mathrm{Fix}_V(g)$ by $\omega'(x,y) = \omega(x,y)$ for all $x, y \in \mathrm{Fix}_V(g)$. Let m be the order of g. For any $x \in \mathrm{Fix}_V(g)$ there is an element $y \in V$ such that $\omega(x,y) = 1$. Define $y' = \sum_{i=0}^{m-1} y^{g^i}$. Clearly, $y' \in \mathrm{Fix}_V(g)$ and $\omega'(x,y') = \sum_{i=0}^{m-1} \omega(x, y^{g^i}) = \sum_{i=0}^{m-1} \omega(x^{g^{-i}}, y) = \sum_{i=0}^{m-1} \omega(x,y) = m$, so that the radical of ω' is trivial, or, in other words, ω' is symplectic. $\square$

The lemma implies that for any $g \neq 1$ the codimension of $\mathrm{Fix}(g)$ in V is greater than 1, since the action of G is faithful. It follows from Theorem 1.1 (iii) that V/G is always singular when G is nontrivial.

We shall assume that the triple (V, ω, G) is *indecomposable*, that is, there exists no ω-orthogonal decomposition $V = V_1 \oplus V_2$ where V_1 and V_2 are symplectic G-stable subspaces. The set

$$S = \{g \in G : \mathrm{Fix}_V(g) \text{ has codimension 2 in } V\}$$

is a subset of G stable under conjugation which we call the set of *symplectic reflections.* For each $s \in S$, V splits as an ω-orthogonal sum $\mathrm{Fix}_V(s) \oplus \mathrm{Im}(\mathrm{Id} - s)$, and we can define an alternating form, ω_s, on V whose radical is $\mathrm{Fix}_V(s)$ and is the restriction of ω on $\mathrm{Im}(\mathrm{Id} - s)$. Let TV denote the tensor algebra on V - the action of G on V induces an action of G on TV.

Definition 1.6. *Let $t \in \mathbb{C}$ and $c : S \to \mathbb{C}$ be a class function, that is, a function constant on the conjugacy classes of S. Then the symplectic reflection algebra, $H_{t,c}$, is the factor algebra of the skew group ring $TV * G$ by the ideal*

$$\left\langle v \otimes w - w \otimes v - t\omega(v,w) - \sum_{s \in S} c(s)\omega_s(v,w)s, \text{ for all } v, w \in V \right\rangle. \tag{1.2}$$

We have defined a whole family of algebras depending on the parameters t and c. When $t = c = 0$ then the relations in (1.2) simply say that v and w commute for $v, w \in V$, thus $H_{0,0} = SV * G$. On the other hand, if $t = 1$ and $c = 0$ then, by choosing a symplectic basis, $\{x_1, \dots, x_n, y_1, \dots, y_n\}$, of V one can see that the relations in (1.2) become those for the nth Weyl algebra, A_n, so that $H_{1,0} = A_n * G$. By Theorem 1.3 (iv), $H_{1,0}$ is a simple ring.

Although the parameter t can be any complex number we can always assume that it equals either 0 or 1, due to the following.

Lemma 1.7. *For any $\lambda \in \mathbb{C}^*$, $H_{t,c} \cong H_{\lambda t, \lambda c}$.*

Proof. If we choose a symplectic basis of $x_i s$ and $y_j s$ as in the previous paragraph then the map $\phi : H_{t,c} \to H_{\lambda t,\lambda c}$ given by $\phi(x_i) = \frac{1}{\sqrt{\lambda}}x_i$, $\phi(y_j) = \frac{1}{\sqrt{\lambda}}y_j$ and $\phi(g) = g$ for all i, j and $g \in G$ is an isomorphism. $\square$

By using associated graded techniques we can deduce some elementary properties of the $H_{t,c}$. Define a nonnegative filtration, $\mathcal{H}$, on $H_{t,c}$ by $\mathcal{H}_0 = \mathbb{C}G$, $\mathcal{H}_1 = \mathbb{C}G + V \otimes \mathbb{C}G$, and $\mathcal{H}_i = (\mathcal{H}_1)^i$ for all $i \geq 2$.

Theorem 1.8. *[26, Theorem 1.3] There is an algebra isomorphism*

$$\operatorname{gr} H_{t,c} \cong SV * G. \tag{1.3}$$

Corollary 1.9. *The algebras $H_{t,c}$ are prime Noetherian rings which are isomorphic as vector spaces to $SV \otimes_{\mathbb{C}} \mathbb{C}G$. They have GK dimension equal to $\operatorname{Dim}_{\mathbb{C}} V$ and global dimension less than or equal to $\operatorname{Dim}_{\mathbb{C}} V$.*

Proof. These are all consequences of the isomorphism (1.3) and the properties of $SV * G$, see [60] for example. $\square$

In practice, one often works with a slightly smaller subalgebra of $H_{t,c}$. Recall the symmetrising idempotent, e, of $\mathbb{C}G$ defined in Section 1.2.

Definition 1.10. *The subalgebra, $eH_{t,c}e$, of $H_{t,c}$ is called the spherical subalgebra of $H_{t,c}$.*

The filtration $\mathcal{H}$ induces a filtration on the subalgebra $eH_{t,c}e$ which we call $\mathcal{E}$.

Corollary 1.11. *The associated graded algebra of $eH_{t,c}e$ is isomorphic to SV^G. Thus $eH_{t,c}e$ is a finitely generated Noetherian domain whose GK dimension equals $\operatorname{Dim}_{\mathbb{C}} V$.*

Proof. Since e lies in the degree zero part of the filtration of $H_{t,c}$ and by Theorem 1.3 and Lemma 1.4 we have

$$\operatorname{gr} eH_{t,c}e = e(\operatorname{gr} H_{t,c})e \cong e(SV * G)e \cong SV^G.$$

Since SV^G is a finitely generated Noetherian domain of GK dimension equal to $\operatorname{Dim}_{\mathbb{C}} V$ (Theorem 1.1) these properties pass up to $eH_{t,c}e$. $\square$

The behaviour of symplectic reflection algebras varies dramatically according to whether the parameter t is equal to one or to zero. This is already evident in the case $c = 0$ discussed above: we have seen that the centre of $H_{0,0} = SV * G$ is SV^G, whereas the centre of $H_{1,0} = A_n * G$ is $\mathbb{C}$ (we could give a direct proof of this, but it follows from the more general result, Theorem 1.13 (ii), below).

Definition 1.12. *Let $Z_{t,c}$ denote the centre of $H_{t,c}$. We define the Satake homomorphism to be the algebra homomorphism*

$$\psi : Z_{t,c} \to eH_{t,c}e; \; z \mapsto ze \text{ for all } z \in Z_{t,c}. \tag{1.4}$$

By an adaptation of the argument of [26, Theorem 3.1] this map is an isomorphism between $Z_{t,c}$ and the centre of $eH_{t,c}e$.

Theorem 1.13.

(i) [26, Theorem 1.6] The algebra $eH_{0,c}e$ is commutative for all c.

(ii) [11, Proposition 7.2] The algebra $eH_{1,c}e$ has centre equal to $\mathbb{C}$ for all c.

Therefore, in the case $t = 0$, $Z_{0,c}$ is isomorphic to $eH_{0,c}e$ via the Satake homomorphism. We note that $Z_{0,0} = \mathcal{O}(V)^G$ and the Satake isomorphism for $t = 0, c = 0$ was proved in Lemma 1.4. The subalgebra $Z_{0,c}$ inherits a filtration (which we call $\mathcal{Z}$) from the one on $H_{0,c}$; it is easy to show that the Satake isomorphism preserves filtrations, that is, $\psi(\mathcal{Z}_i) = \mathcal{E}_i$ for all $i \in \mathbb{Z}$, because e lies in degree zero. We have the following.

Proposition 1.14. *[26, Theorem 3.3 and Proposition 3.4] There are algebra isomorphisms*

$$\operatorname{gr}^{\mathcal{Z}} Z_{0,c} \cong \mathcal{O}(V)^G,$$

and

$$\operatorname{gr}\psi : \operatorname{gr}^{\mathcal{Z}} Z_{0,c} \to \operatorname{gr}^{\mathcal{E}} eH_{0,c}e,$$

where $\operatorname{gr}\psi$ is the associated graded map of ψ.

Corollary 1.15. *For all c, $H_{0,c}$ is finite over $Z_{0,c}$.*

Proof. We have $Z_{0,c} \subset H_{0,c}$ and the filtration $\mathcal{Z}$ is induced from $\mathcal{H}$. Now $\mathrm{gr}^{\mathcal{Z}} Z_{0,c} \cong \mathcal{O}(V)^G \subset \mathcal{O}(V) * G \cong \mathrm{gr}^{\mathcal{H}} H_{0,c}$, and $\mathcal{O}(V) * G$ is finite over $\mathcal{O}(V)^G$ by Lemma 1.4. This property now lifts to $H_{0,c}$ and $Z_{0,c}$ by standard associated graded arguments, [60]. $\square$

The above corollary implies that $H_{0,c}$ is a *polynomial identity algebra*, or PI algebra for short (see [67] or [60] for the general theory of PI algebras). In general any prime PI algebra has a nontrivial centre, [60, Theorem 13.6.4], and is often finite over its centre (for instance if the centre is Noetherian, [60, Proposition 13.6.11]). Theorem 1.13 tells us that the centre of $H_{1,c}$ is trivial for all c so that $H_{1,c}$ is certainly not a PI algebra.

It is here that the theory of symplectic reflection algebras splits into two distinct paths. Our interest is the study of the representation theory of the $H_{0,c}$, where the geometry of the centres plays an important role and there are many finite dimensional representations. The representation theory of $H_{1,c}$ is quite different, see [32], [30] and [34]. In particular, the $t = 1$ case has connections with differential operators and finite dimensional representations are much rarer, [6]. As an indication of this, the algebras $H_{1,0}$ are skew group algebras of Weyl algebras; they are simple and so have no finite dimensional representations.

Notation 1.16. For the remainder of the thesis we shall write $H_c = H_{0,c}$, $Z_c = Z_{0,c}$ and $X_c = \mathrm{Max}\, Z_c$.

We note that a consequence of Proposition 1.14 is that Z_c is a domain and is finitely generated as a $\mathbb{C}$-algebra. Therefore by Theorem 1.1 (iv) the variety X_c is irreducible of dimension $\mathrm{Dim}_{\mathbb{C}} V$.

Let irr H_c denote the the set of simple H_c-modules. All simple H_c-modules are finite dimensional because H_c is an affine PI algebra, [60, Theorem 13.10.7 (i)]. Let $M \in \mathrm{irr}\ H_c$. It follows from Schur's Lemma that Z_c acts by scalars on M and therefore that $\mathrm{ann}_{Z_c} M$ is a maximal ideal of Z_c by the Nullstellensatz. Thus we have the following map

$$\chi : \mathrm{irr}\ H_c \to X_c;\ M \mapsto \mathrm{ann}_{Z_c} M. \tag{1.5}$$

For any point $\mathfrak{m} \in X_c$ we have

$$\chi^{-1}(\mathfrak{m}) = \mathrm{irr}\ H_c/\mathfrak{m}H_c$$

where the factor algebra $H_c/\mathfrak{m}H_c$ is a finite dimensional algebra by Corollary 1.15. Let $N = \mathrm{Max}\{\mathrm{Dim}_{\mathbb{C}} M : M \in \mathrm{irr}\ H_c\}$. By [10, Theorem I.13.5(i)] N is finite and we call this number the

PI degree of H_c. The *Azumaya locus*, $\mathcal{A}$, of X_c is the set of maximal ideals, $\mathfrak{m}$, such that $\chi^{-1}(\mathfrak{m})$ contains a module of dimension N. If $\mathfrak{m} \in \mathcal{A}$ then $\chi^{-1}(\mathfrak{m})$ is a singleton and $H_c/\mathfrak{m}H_c \cong \mathrm{Mat}_{N\times N}(\mathbb{C})$ by the Artin-Procesi Theorem [60, Theorem 13.7.14].

Theorem 1.17. *The Azumaya locus of X_c, $\mathcal{A}_c$, equals the smooth locus and for any point $\mathfrak{m}$ in the Azumaya locus, the unique simple module in $\chi^{-1}(\mathfrak{m})$ is isomorphic, as a G-module, to the regular representation of G.*

Proof. By [26, Theorem 1.7], for any point $\mathfrak{m} \in \mathrm{sm}X_c$ there is a unique simple module in $\chi^{-1}(\mathfrak{m})$ which is isomorphic to the regular representation of G. By [9, Lemma 3.3], $\mathcal{A}_c \subseteq \mathrm{sm}X_c$ and so by definition of the Azumaya locus we must have $\mathcal{A}_c = \mathrm{sm}X_c$. □

Clearly it is interesting to know when X_c is singular, and thus when H_c has small simple modules (that is, whose dimension is less than the PI degree). Little is known about the nature of X_c, but there are some results for certain classes of triples (V, ω, G) which we describe in the next section. In fact there is a natural Poisson structure on X_c which allows one to make more precise the relationship between the geometry of the centres and the representation theory of symplectic reflection algebras. We describe this further in Section 1.6.

1.4 Classes of symplectic reflection algebras

In this section we shall consider the two major examples of symplectic reflection algebras. We begin with a reduction procedure which will allow to restrict our attention to groups generated by symplectic reflections. Let (V, ω, G) be an indecomposable triple with corresponding symplectic reflection algebra $H_{t,c}$. The subgroup, H, of G, generated by the symplectic reflections, S, is a normal subgroup since S is stable under conjugation. However, the triple (V, ω, H) need not be indecomposable.

Lemma 1.18. *If V_1 is an H-invariant symplectic subspace of V then there is a ω-orthogonal decomposition $V = V_1 \oplus V_2$ such that $H = H_1 \times H_2$ with $H_i \in Sp(V_i)$, and each H_i is generated by symplectic reflections on V_i.*

Proof. Set $V_2 = \{x \in V : \omega(x, v) = 0 \text{ for all } v \in V_1\}$. Since V_1 is symplectic, $V = V_1 \oplus V_2$ and V_2 is also symplectic. For all $v_1 \in V_1$, $v_2 \in V_2$ and $g \in H$

$$\omega(v_1, v_2^g) = \omega(v_1^{g^{-1}}, v_2) = 0$$

by the H-invariance of ω and V_1. Therefore V_2 is H-invariant also.

The above paragraph implies that $H \leq Sp(V_1) \times Sp(V_2)$. Let H_i be the image of H on $Sp(V_i)$ so that $H \leq H_1 \times H_2$. Let $g \in H$ be a symplectic reflection; clearly

$$\mathrm{Fix}_V(g) = \mathrm{Fix}_{V_1}(g) \oplus \mathrm{Fix}_{V_2}(g).$$

By Lemma 1.5, $\mathrm{Fix}_{V_1}(g)$ and $\mathrm{Fix}_{V_2}(g)$ are symplectic subspaces of V_1 and V_2 respectively. Therefore, because g is a symplectic reflection, it acts trivially on one summand and as a symplectic reflection on the other. Since H is generated by symplectic reflections, both of H_1 and H_2 are too, and H contains the symplectic reflections of both H_1 and H_2 so $H = H_1 \times H_2$. $\square$

Now consider a maximal ω-orthogonal decomposition $V = V_1 \oplus \cdots \oplus V_k$ such that H preserves each summand. By the lemma we can write $H = H_1 \times \cdots \times H_k$ where $H_i \in Sp(V_i)$ is generated by symplectic reflections. Let ω_i be the symplectic form on V_i. Then, by maximality of the decomposition, each triple (V_i, ω_i, H_i) is indecomposable. Let $c_i : S \cap H_i \to \mathbb{C}$ be the restriction of c to $S \cap H_i$. We denote the symplectic reflection algebra for (V_i, ω_i, H_i) at parameter c_i by H_{c_i}. The crossed product is defined in [64] and is a generalisation of the skew group ring.

Theorem 1.19. *There is an algebra isomorphism*

$$H_c \cong (H_{c_1} \otimes \cdots \otimes H_{c_k}) * G/H$$

where the latter algebra is the crossed product of G/H with coefficients in $(H_{c_1} \otimes \cdots \otimes H_{c_k})$.

Proof. It follows from the proof of Lemma 1.18 that if $i \neq j$ then for all $s \in S, v_i \in V_i$ and $v_j \in V_j$, $\omega_s(v_i, v_j) = 0$ and so $v_i, v_j \in H_c$ commute. Furthermore, if $i = j$ and $s \notin H_i$ then $\omega_s(v_i, v_j) = 0$. Thus the subalgebra of H_c generated by $V = V_1 \oplus \cdots \oplus V_k$ and $H = H_1 \times \cdots \times H_k$ is isomorphic to $H_{c_1} \otimes \cdots \otimes H_{c_k}$. Denote this subalgebra by K. Choose coset representatives, $g_1 = 1, \ldots, g_t$, of G/H. Then $H_c = \bigoplus_{i=1}^t K g_i$ as a K-module and $g_i K = K g_i$ for all i since H is normal in G. Thus $H_c \cong (H_{c_1} \otimes \cdots \otimes H_{c_k}) * G/H$. $\square$

Therefore, by Theorem 1.19, it is natural to reduce the study of all symplectic reflection algebras to those where one has an indecomposable triple (V, ω, G) where G is generated by symplectic reflections. There is a classification of all such triples which is a consequence of results in [40], see also [37. Theorems 10.1 and 10.2].

Theorem 1.20. *Let (V, ω, G) be an indecomposable symplectic triple such that G is generated by symplectic reflections on V.*

1. *If V is not an irreducible G-module then $V = U \oplus U^*$ where U is an irreducible G-module, and $G \subseteq \mathrm{GL}(U)$ is generated by complex reflections.*
2. *If V is an irreducible G-module then we have two possibilities.*
   - *(i) $V = V_1 \oplus \cdots \oplus V_n$ is a ω-orthogonal decomposition where each V_i is a 2-dimensional symplectic subspace of V and G permutes the V_i as S_n. Then G is a subgroup of the wreath product $(H \times \cdots \times H) \rtimes S_n$ where $H \leq Sp(V_1)$ is the image of the stabiliser of V_1 in $GL(V_1)$.*
   - *(ii)* $\mathrm{Dim}_{\mathbb{C}} V \leq 10$ *and G is one of a finite list of groups given in [37, Theorem 10.2].*

There are two families of triples which occur naturally in this classification and we consider these in turn.

The reducible case

Suppose that the triple (V, ω, G) is reducible so that $V = U \oplus U^*$ where U is an irreducible G-module, and $G \subseteq \mathrm{GL}(U)$ is generated by complex reflections. The symplectic form, ω, can be described as follows. For any $(u, \alpha), (v, \beta) \in U \oplus U^* = V$,

$$\omega((u, \alpha), (v, \beta)) = \alpha(v) - \beta(u).$$

The finite irreducible subgroups of $\mathrm{GL}_n(\mathbb{C})$ generated by complex reflections have been classified by Shephard and Todd in [72] into 34 classes. These include the finite Coxeter groups (see [41]) and so, in particular, include the case where U is the Cartan subalgebra of a finite dimensional simple complex Lie algebra and G is the corresponding Weyl group. The symplectic reflection algebras for triples in the latter case are often called *rational Cherednik algebras.*

Theorem 1.21. *[32, Proposition 7.3], [26, Theorem 16.4] Let G be a Coxeter group and U its reflection representation. The variety X_c is singular for all values of c if G is of type D_{2n}, E, F, G, H or $I_2(m)$ for $m \geq 5$.*

Therefore in all the examples listed above the symplectic reflection algebras have small representations (in fact Theorem 1.21 is proved for many types by establishing the existence of such representations). The cases where G is of type A or B are discussed in the next section (the Weyl groups of type C are the same as those of type B).

Wreath products

We shall return to the case of wreath products later on so it will be useful to fix some notation.

Notation 1.22. Let L be a two dimensional vector space with symplectic form ω_L. We fix a symplectic basis of L, x, y. For any $n \geq 1$ we can form a symplectic vector space $V = L^{\oplus n} = L \oplus \cdots \oplus L$. For any $v \in V$ we shall write v_i for the element $(0, \ldots, v, \ldots, 0)$ with v in the ith position. There is a symplectic form on V given by $\omega = \omega_L \oplus \cdots \oplus \omega_L$. Let Γ be any finite subgroup of $SL(L)$. Then $\Gamma \times \cdots \times \Gamma$ acts on V and so does S_n, the symmetric group on n letters, by permuting the copies of L. Both of these actions preserve ω. Thus the semidirect product $S_n \ltimes (\Gamma \times \cdots \times \Gamma)$ acts on V preserving the form. We refer to this group as a wreath product and denote it by Γ_n.

We list the symplectic reflections of Γ_n. Given $\gamma \in \Gamma$, we write $\gamma_i \in \Gamma_n$ for the element γ regarded as an element of the i-th factor Γ. Let $s_{ij} \in S_n$ denote the transposition swapping i and j. Then the group Γ_n is generated by the symplectic reflections s_{ij} and γ_i, where $\gamma \neq 1$. The conjugacy classes of symplectic reflections in $\Gamma_n < Sp(V)$ are known to be of two types:

1. The set $\{s_{ij} \cdot \gamma_i \cdot \gamma_j^{-1} : i, j \in [1, n], i \neq j, \gamma \in \Gamma\}$ forms a single Γ_n conjugacy class;
2. Elements $\{\gamma_i : i \in [1, n], \gamma \in \mathcal{C}\}$ form one Γ_n conjugacy class for any given conjugacy class, $\mathcal{C}$, in $\Gamma \setminus \{\mathrm{Id}\}$.

In particular, we can identify the set of class functions $S \to \mathbb{C}$ with $\mathbb{C}^{k+1}$ where $k+1$ is the number of conjugacy classes in Γ. Therefore the flat family of (centres of) symplectic reflection algebras is parametrised by elements $c \in \mathbb{C}^{k+1}$. We will assume that $c = (c_1, \underline{c})$ where $c_1 \in \mathbb{C}$ corresponds to the conjugacy class (1) above and $\underline{c} \in \mathbb{C}^k$ corresponds to class functions $\Gamma \setminus \{1\} \to \mathbb{C}$.

An easy calculation shows that $Sp(L) = SL(L)$, and the finite subgroups of $SL(L)$ are known and have been classified by certain graphs (called McKay graphs) which are the extended Dynkin diagrams of type $\tilde{A}, \tilde{D}$ and $\tilde{E}$, see [14]. The McKay graph of Γ is obtained as follows. Let $S_0, \ldots, S_k$

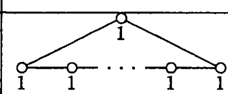
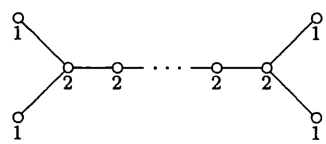
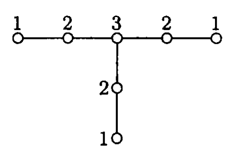
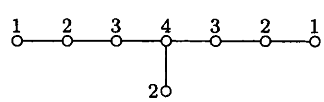
be the irreducible representations of Γ with S_0 being the trivial representation. Now form the graph with vertex set $I = \{0, \ldots, k\}$ and the number of edges between i and j is the multiplicity of S_i in $S_j \otimes L$.

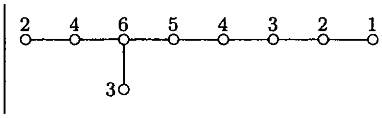
We give an explicit description of the finite subgroups of $SL(L)$ below. We write them with respect to the basis $\{x, y\}$ and give their orders and their McKay graphs. Let $\eta_n = e^{\frac{2\pi i}{n}}$. We introduce the following elements of $SL(L)$:

$$\rho_n = \begin{pmatrix} \eta_n & 0 \\ 0 & \eta_n^{-1} \end{pmatrix}, \alpha = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \gamma = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta_8^7 & \eta_8^7 \\ \eta_8^5 & \eta_8 \end{pmatrix}$$

$$\varepsilon = \begin{pmatrix} -\eta_5^3 & 0 \\ 0 & -\eta_5^2 \end{pmatrix}, \zeta = \frac{1}{\eta_5^2 - \eta_5^{-2}} \begin{pmatrix} \eta_5 + \eta_5^{-1} & 1 \\ 1 & -(\eta_5 + \eta_5^{-1}) \end{pmatrix}.$$

We shall label the vertices of each graph with the dimension of the corresponding irreducible representation.

Type	Group	Order	Value of $k+1$	McKay Graph
$\tilde{A}_n$ $(n \geq 1)$	$\langle \rho_n \rangle$	n	n	
$\tilde{D}_n$ $(n \geq 2)$	$\langle \rho_{2n},\ \alpha \rangle$	4n	n+3	
$\tilde{E}_6$	$\langle \rho_4,\ \alpha,\ \gamma \rangle$	24	7	
$\tilde{E}_7$	$\langle \rho_8,\ \alpha,\ \gamma \rangle$	48	8	

$\tilde{E}_8$	$\langle\beta,\ \varepsilon,\ \zeta\rangle$	120	9	

Symplectic reflection algebras for wreath products are usually Azumaya over their centres - this follows from the following theorem and Theorem 1.17. One of our main results, Corollary 4.25, is to use the methods of the next theorem to give a concrete description of X_c for wreath products for all values of c.

Theorem 1.23. *[26, Corollary 1.14(i)] For any finite subgroup $\Gamma \subseteq SL(L)$ and any $n \geq 1$, the variety X_c associated to Γ_n is smooth for generic values of c.*

There is a certain overlap between the reducible case and wreath products. Suppose that Γ is of type $\tilde{A}$ as listed above. Consider the basis of V, $\{x_i, y_j : 1 \leq i, j \leq n\}$. The subspace, U, spanned by all x_i is stable under Γ_n. If Γ is nontrivial then one calculates that U is an irreducible G-module. The representation $G \subseteq GL(U)$ allows us to identify G with a complex reflection group (by comparing with the list in [72] - here G is the group $G(m,1,n)$ where m is the order of Γ). The subspace spanned by the y_i can be identified with U^* via the form, ω, and so these triples all fall into case (1) of Theorem 1.20. In particular, the groups, $G(2,1,n)$ are the Weyl groups of type B. We get as a result of this and Theorem 1.23.

Corollary 1.24. *When G is the Weyl group of type B and U is its reflection representation then the variety, X_c, corresponding to the action of G on $V = U \oplus U^*$ is smooth for generic values of c.*

When Γ is the trivial group one can still form the subspace U of V, and it is G-stable, but it is not an irreducible G-module. In fact, $U = \mathfrak{h} \oplus T$ where $\mathfrak{h}$ is the reflection representation of $G = S_n$, and T is the trivial representation. Then G acts by symplectic reflections on $\mathfrak{h} \oplus \mathfrak{h}^*$ and V decomposes as a ω-orthogonal sum $(\mathfrak{h} \oplus \mathfrak{h}^*) \oplus (T \oplus T^*)$ and this decomposition is maximal by Theorem 1.20. Write $G = H_1 \times H_2$ as in Lemma 1.18 (in this rather degenerate case, $H_1 = G$ and H_2 is trivial). Then by Theorem 1.19,

$$H_c = H_{c_1} \otimes H_{c_2}.$$

However, it is clear that H_{c_2} is simply a polynomial algebra in two variables. Hence we have

$$X_c \cong X_{c_1} \times \mathbb{C}^2. \tag{1.6}$$

The variety X_{c_1} is the variety associated to the centre of a rational Cherednik algebra of type A_{n-1}.

Proposition 1.25. *The variety X_{c_1} is smooth for all nonzero values of c_1.*

Proof. It is known that there is a single conjugacy class of symplectic reflections in $H_1 = S_n$, see Notation 1.22, so c_1 is just a complex number. By Lemma 1.7, $X_{c_1} \cong X_{\lambda c_1}$ for all $\lambda \in \mathbb{C}^*$. By Theorem 1.23 and (1.6) it follows that X_{c_1} is smooth for all $c_1 \neq 0$. □

1.5 Poisson algebras

In this section we introduce Poisson algebras, which turn out to play an important role in the study of symplectic reflection algebras.

Definition 1.26. *Let R be a commutative $\mathbb{C}$-algebra. Then R is a Poisson algebra if there exists a non-trivial Poisson bracket $\{-,-\} : R \times R \to R$. That is, $\{-,-\}$ is a Lie bracket and for all $x, y, z \in R$, $\{xy, z\} = x\{y, z\} + \{x, z\}y$.*

Let R be a Poisson algebra. A *Poisson ideal* is an ideal, I, of R such that $\{R, I\} \subseteq I$. By [23, 3.3.2] the radical of any Poisson ideal is Poisson. Let $(R, \{-,-\}_R)$ and $(S, \{-,-\}_S)$ be Poisson algebras. We say that a map $\psi : R \to S$ is a *Poisson homomorphism* if it is an algebra homomorphism such that for all $x, y \in R$, $\psi(\{x,y\}_R) = \{\psi(x), \psi(y)\}_S$.

Let $(R, \{-,-\})$ be a Poisson algebra and $\mathcal{M}$ a multiplicatively closed subset. Then there is a Poisson bracket on the localisation, $R_{\mathcal{M}}$, given by the quotient rule making the natural homomorphism $R \to R_{\mathcal{M}}$ a Poisson homomorphism. Factor rings of R are less well behaved when it comes to Poisson structures, but if I is a Poisson ideal of R then R/I is a Poisson algebra with bracket $\{r+I, s+I\} = \{r, s\} + I$. Given two Poisson algebras $(R, \{-,-\}_R)$ and $(S, \{-,-\}_S)$ we can define a Poisson bracket on the tensor product $R \otimes_{\mathbb{C}} S$ by $\{r \otimes s, r' \otimes s'\} = \{r, r'\}_R \otimes ss' + rr' \otimes \{s, s'\}_S$ for all $r, r' \in R$, $s, s' \in S$. This makes the canonical maps $R, S \to R \otimes_{\mathbb{C}} S$ into Poisson homomorphisms.

When $R = \bigoplus_{i=0}^{\infty} R_i$ is a graded Poisson algebra we shall say that $\{-,-\}$ *has degree l* if for all i, j, $\{R_i, R_j\} \subseteq R_{i+j+l}$, and l is the minimal integer for which this is true. Similarly, when $R = \bigcup_{i=0}^{\infty} R_i$ is a filtered Poisson algebra we shall say that $\{-,-\}$ *has degree l* if for all i, j, $\{R_i, R_j\} \subseteq R_{i+j+l}$, and l is the minimal integer for which this is true.

Example 1.27. Let (V, ω) be a symplectic vector space and let $G \subseteq Sp(V)$ be a subgroup. Then SV^G is a Poisson algebra. The symplectic form ω induces a Poisson bracket on SV: by the Leibniz rule it suffices to define the bracket on algebra generators of SV and in the symplectic basis of $x_i s$ and $y_j s$ introduced in the previous section this bracket is $\{x_i, x_j\} = \{y_i, y_j\} = 0$ for all i, j and $\{x_i, y_j\} = \delta_{ij}$. It is clear from the definition that this bracket has degree -2. Now the fact that G preserves the form ω means that this Poisson bracket restricts to a Poisson bracket, $\{-,-\}_\omega$, on the subalgebra SV^G, and $\{-,-\}_\omega$ has degree -2 with respect to the induced graded structure on SV^G.

Example 1.28. Let $\mathfrak{g}$ be a complex Lie algebra. Then the symmetric algebra on $\mathfrak{g}$, $S\mathfrak{g}$, has a Poisson bracket obtained by specifying that $\{x, y\} = [x, y]$ for all $x, y \in \mathfrak{g}$. This bracket has degree -1 and is often referred to as the Kostant-Kirillov-Souriau bracket.

Let R be a Poisson algebra and N an R-module. Then we say that N is a *Poisson module* if there exists a $\mathbb{C}$-bilinear map $\{-,-\}_N : R \times N \to N$ satisfying $\{r, r'n\}_N = \{r, r'\}n + r'\{r, n\}_N$ for all $r, r' \in R$ and $n \in N$.

Recall that for a commutative ring R and an R-module N the *associated primes* of N, written $\mathrm{Ass}N$, are the set of primes of R which are annihilators of elements of N. The next lemma will come in handy.

Lemma 1.29. *[13, Theorem 4.5] Let R be a Noetherian Poisson algebra and N be a finitely generated Poisson R-module. Then the associated primes of N are Poisson ideals of R.*

Proof. Let P be an associated prime of N. Let $L = \{n \in N : P^i n = 0 \text{ for some } i \geq 0\}$. This is a non-zero submodule of N. We claim that L is a Poisson submodule of N. To show this we first note that since L is finitely generated there is some $t \geq 1$ such that $P^t L = 0$. Let $l \in L, r \in R$. For all $r' \in P^t$,

$$0 = \{r, r'l\}_N = \{r, r'\}l + r'\{r, l\}_N.$$

Therefore $P^t\{R, L\}_N \subseteq \{R, P^t\}L \subseteq L$, which implies that $P^{2t}\{R, L\}_N = 0$. Hence $\{R, L\}_N \subseteq L$, by definition of L, which means that L is a Poisson submodule of N. By [11, Lemma 4.1], $\mathfrak{J} = \mathrm{ann}_R L$ is a Poisson ideal of R. There is some element $x \in L$ such that $\mathrm{Ann}_R\{x\} = P$, so $x \in L$ implies $\mathfrak{J} \subseteq P$. Taking radicals of the ideals $P^t \subseteq \mathfrak{J} \subseteq P$ yields $\mathrm{rad}\mathfrak{J} = P$, and therefore P is a Poisson ideal by [23, 3.3.2]. □

We describe a method for obtaining a Poisson algebra from a family of noncommutative algebras, and we apply this to symplectic reflection algebras.

Lemma 1.30 (Quantisation). *Let $\tilde{A}$ be a free $\mathbb{C}[t]$-algebra and suppose that Z is an affine central commutative subalgebra of $\tilde{A}/t\tilde{A}$. Let $\pi : \tilde{A} \to \tilde{A}/t\tilde{A}$ be the quotient map. Let $\tilde{Z} = \pi^{-1}(Z)$ which is a subalgebra of $\tilde{A}$. For $a, b \in Z$ choose lifts $\tilde{a}, \tilde{b} \in \tilde{Z}$. Then the bracket*

$$\{a, b\} := \pi([\tilde{a}, \tilde{b}]/t) \tag{1.7}$$

is well-defined. Furthermore, if $\{-,-\}$ is nonzero then it defines a Poisson bracket on Z.

Proof. The right hand side of (1.7) makes sense because Z is commutative so that $[x, y] \in t\tilde{A}$ for all $x, y \in \tilde{A}$. The bracket is well-defined because Z is central in $\tilde{A}/t\tilde{A}$. It is standard to check that if $\{-,-\}$ is nonzero then it yields a Poisson bracket on Z. □

In the case of symplectic reflection algebras we can take t to be an indeterminate in Definition 1.6. We denote this $\mathbb{C}[t]$-algebra by $\hat{H}_{t,c}$. On the one hand $\hat{H}_{t,c}/t\hat{H}_{t,c} \cong H_c$ has centre Z_c and so this obtains a bracket (which is possibly zero) by the lemma. On the other hand if we take $\tilde{A} = e\hat{H}_{t,c}e$ then the quotient $\tilde{A}/t\tilde{A} \cong eH_ce$ is commutative by Theorem 1.13 and so the spherical subalgebra obtains a bracket via quantisation also.

Proposition 1.31. *[26, Claim 2.25(i) and Theorem 3.1] The brackets on Z_c and eH_ce obtained by quantisation are nontrivial. Furthermore, the Satake isomorphism is a Poisson isomorphism.*

We will therefore refer to the Poisson bracket obtained on either eH_ce or Z_c by $\{-,-\}$. The pair (H_c, Z_c) is a *Poisson order* in the sense of [11], which has interesting consequences for the representation theory of H_c, as we shall see later.

The algebra $Z_0 = SV^G$ has two Poisson structures: one, $\{-,-\}$, coming from quantisation, and one, $\{-,-\}_\omega$, coming from the form ω (see Example 1.27).

Lemma 1.32. *The Poisson brackets $\{-,-\}$ and $\{-,-\}_\omega$ on SV^G are equal.*

Proof. We can write $\hat{H}_{t,0}$ as the $\mathbb{C}[t]$-algebra

$$\mathbb{C}[t]\langle x_1, \dots, x_n, y_1, \dots, y_n : [x_i, x_j] = [y_i, y_j] = 0, [x_i, y_j] = \delta_{ij}t \text{ for all } i, j\rangle * G.$$

Let $A \subset \hat{H}_{t,0}$ be the subalgebra generated by the elements $x_1, \dots, x_n, y_1, \dots, y_n$. Using the fact that $\hat{H}_{t,0}$ is a free $\mathbb{C}[t]$-module we can calculate the bracket $\{-,-\}$ by performing quantisation in A. Clearly $A/tA \cong SV$ and the brackets obtained on SV by quantisation and via ω are equal. The action of G on V extends to an action on A which fixes t. The map $A^G \hookrightarrow A$ induces an isomorphism $(A/tA)^G \cong A^G/tA^G$ because we are working over field of characteristic zero and so have a Reynolds operator. Therefore $A^G/tA^G \cong SV^G$. Let $\pi : A \to SV$ and $\pi' : A^G \to A^G/tA^G \cong SV^G$ be the quotient maps. The following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\pi"] & SV \\ A^G \arrow[u] \arrow[r, "\pi'"'] & SV^G. \arrow[u] \end{tikzcd}$$

Let $a, b \in SV^G$. Choose lifts $\tilde{a}, \tilde{b} \in A$. We can assume that $\tilde{a}, \tilde{b} \in A^G$. Then $\{a, b\} = \pi'([\tilde{a}, \tilde{b}]/t) = \pi([\tilde{a}, \tilde{b}]/t) = \{a, b\}_\omega$. $\square$

We shall refer to this bracket on SV^G by $\{-,-\}_\omega$.

1.6 Poisson geometry and symplectic leaves

Poisson varieties

We shall say that an affine algebraic variety X is *Poisson* if $\mathcal{O}(X)$ is a Poisson algebra. Thus the variety corresponding to SV^G, the orbit space V/G, is a Poisson variety; for any finite dimensional complex Lie algebra, $\mathfrak{g}$, $S\mathfrak{g}$ is Poisson so that $\mathfrak{g}^*$ is a Poisson variety. In general, given finitely generated Poisson algebras A and B, then $\operatorname{Max} A$, $\operatorname{Max} B$ and $\operatorname{Max} A \times \operatorname{Max} B$ are Poisson varieties. A morphism, $\phi : X \to Y$, between two Poisson varieties, X and Y, is called *Poisson* if the corresponding comorphism $\phi^\sharp : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a Poisson homomorphism.

A *Poisson subvariety* of a Poisson variety, X, is a affine subvariety, Y, which is Poisson such that the morphism $Y \to X$ is Poisson. For a Poisson algebra, A, and any Poisson ideal, I, of A, $\operatorname{Max} A/I$ is a closed Poisson subvariety of $\operatorname{Max} A$; for any $f \in A$ which is not nilpotent, the standard open subset $\operatorname{Max} A_f \subseteq \operatorname{Max} A$ is an open Poisson subvariety.

Remark 1.33. For a general variety, Z, which is not necessarily affine, one should say that Z is Poisson if $\mathcal{O}_Z$ defines a sheaf of algebras such that the restriction maps are Poisson homomorphisms. For an affine Poisson variety, X, the Poisson bracket on $\mathcal{O}_X(X)$ induces a Poisson structure on the sheaf $\mathcal{O}_X$. Therefore any open subvariety of X is a Poisson variety.

For any $f \in \mathcal{O}(X)$ the *Hamiltonian vector field of f* is the derivation, $\{f, -\}$, of $\mathcal{O}(X)$. When X is smooth and affine one can identify this with the algebraic vector field $\Xi_f : X \to TX$ where $\Xi_f(x) = \{f, -\}(x) \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(X)_x, \mathcal{O}(X)_x/m_x)$.

One common way to obtain Poisson varieties is to induce a Poisson structure from a symplectic one. An affine algebraic variety M is called a *symplectic variety* if it is smooth and comes equipped with a nondegenerate closed algebraic 2-form ω. Here, closed means that $d\omega = 0$ where d is the exterior derivative; ω being nondegenerate means that ω_m is nondegenerate for all $m \in M$ (that is, ω_m is a symplectic form on T_mM). A symplectic variety (M, ω) gives rise to a Poisson bracket on $\mathcal{O}(M)$ as follows. We denote by $\mathrm{Vect}(M)$ the Lie algebra of vector fields on M. Nondegeneracy of ω means that there is a $\mathbb{C}$-linear map $\mathcal{O}(M) \to \mathrm{Vect}(M); f \mapsto \Xi_f$ where Ξ_f is the unique vector field such that $\omega(\Xi_f, -) = df$. For all $f, g \in \mathcal{O}(M)$ we define the bracket of f and g by $\{f, g\} = \Xi_f g = \omega(\Xi_f, \Xi_g)$. This is a Poisson bracket by [15, Theorem 1.2.7], and in this situation $\Xi_f = \{f, -\}$ is the Hamiltonian vector field of f.

Poisson manifolds

Similarly, a manifold, M, is called a *Poisson manifold* if $\mathcal{O}^{hol}(M)$ is a Poisson algebra. A smooth map between Poisson manifolds, $\psi : M \to N$ is called *Poisson* if the corresponding homomorphism $\psi^* : \mathcal{O}^{hol}(N) \to \mathcal{O}^{hol}(M)$ is Poisson. For any $f \in \mathcal{O}^{hol}(M)$ the *Hamiltonian vector field of f* is the derivation, $\{f, -\}$, of $\mathcal{O}^{hol}(M)$. There is a unique vector field $\Xi_f : M \to TM$ so that $\Xi_f \cdot g = \{f, g\}$ for all $g \in \mathcal{O}^{hol}(M)$.

Remark 1.34. Let $\mathbb{P}^n$ be complex projective n-space. Then by Louiville's Theorem $\mathcal{O}^{hol}_{\mathbb{P}^n}(\mathbb{P}^n) = \mathbb{C}$ because $\mathbb{P}^n$ is compact. Thus $\mathbb{P}^n$ has no nontrivial Poisson structure in the sense defined above. One can, however, make $\mathcal{O}^{hol}_{\mathbb{P}^n}$ into a sheaf of Poisson algebras in a nontrivial way. Therefore one might expect that the correct definition of a Poisson manifold should be in terms of sheaves c.f. Remark 1.33. We use our more restrictive definition for two reasons. Firstly we are interested in determining symplectic leaves for Poisson manifolds and there seems to be no 'sheafified' version of Theorem 1.36. Our second reason is that we are interested in calculating symplectic leaves of Poisson varieties. Thus we are led to consider varieties which are quasiaffine and these are compact only if they are finite sets.

Let X be an affine smooth Poisson variety. The Poisson bracket on A is an element of $\bigwedge^2 \mathrm{Der}_{\mathbb{C}} A$.

Now, just as $\mathrm{Der}_{\mathbb{C}} A$ is identified with the space of algebraic vector fields, $\bigwedge^2 \mathrm{Der}_{\mathbb{C}} A$ is isomorphic to the space of algebraic 2-vector fields. The elements of $\bigwedge^2 \mathrm{Der}_{\mathbb{C}} A$ which satisfy the Jacobi identity are precisely the sections $\theta : X \longrightarrow T^2 X$ such that $[\theta, \theta] = 0$, where $[-,-]$ denotes the Schouten bracket; for a definition of the latter see [55, Theorem 10.8.1]. We sometimes call the 2-vector field associated to the bracket the Poisson bivector.

In a similar fashion, a Poisson bracket on a manifold, M, is equivalent to the existence of a holomorphic 2-vector field $\theta : M \longrightarrow \bigwedge^2 TM$ which satisfies $[\theta, \theta] = 0$. Therefore if we begin with an affine algebraic Poisson smooth variety, X, we can induce the structure of a Poisson manifold by considering the Poisson bivector θ associated to the bracket on A: viewing θ as a holomorphic map yields a Poisson bracket on $\mathcal{O}^{hol}(X)$.

If X is not smooth then the Poisson bracket on $\mathcal{O}(X)$ defines a sheaf of Poisson algebras on X. Let U_0 be the smooth locus of X. Each affine open subvariety U in U_0 has a Poisson bracket and so there exists a Poisson bivector $\theta_U : U \longrightarrow T^2 U$. Gluing together these bivectors for all U gives a Poisson bivector $\theta : U_0 \longrightarrow T^2 U_0$ which induces the Poisson bracket on $\mathcal{O}(U_0)$. Thus U_0 becomes a Poisson manifold as in the previous paragraph.

Symplectic manifolds are defined completely analogously to symplectic varieties (except that the 2-form, ω, needs only to be holomorphic). One can mimic the argument for the algebraic case to show that symplectic manifolds are Poisson manifolds.

Definition 1.35. *Let M be a Poisson manifold. The symplectic leaf $\mathcal{S}(p)$ containing a point p of M is the set of points q which are connected to p by piecewise smooth curves, each segment of which is the integral curve of a Hamiltonian vector field.*

Therefore to find the symplectic leaf containing p one works out all the points one can reach by travelling along integral curves of Hamiltonian vector fields at p. For each point q connected to p in this way one repeats the process, finding the points one can reach by travelling along integral curves of Hamiltonian vector fields at q. Then one continues this process until the symplectic leaf is swept out.

The *rank* of p is equal to the dimension of the subspace of $T_p M$ spanned by the Hamiltonian vector fields evaluated at p. For Poisson manifolds M and N such that $N \subseteq M$ we shall say that N is a *Poisson submanifold* of M if the inclusion map is Poisson.

The proof of the following theorem for real manifolds, under some additional hypotheses, goes back to Lie, and in the general case is due to Kirillov, [44]; it is also true in the complex case though, see [2, Theorem 2.1]. Further discussion about symplectic leaves can be found in [55, Section 10] and in [79].

Theorem 1.36. *M is a disjoint union of its symplectic leaves. Each leaf is a symplectic manifold which is a Poisson submanifold of X and the dimension of the leaf through p is equal to the rank of p in M.*

Suppose that X is a smooth affine Poisson variety. Then, as described above, X is a Poisson manifold so one can stratify X by symplectic leaves. Although X is an algebraic variety it is quite possible that the leaves are not varieties, see [11, Remarks 3.6 (1)].

Now suppose that X is not necessarily smooth; we can stratify X by symplectic leaves, as described in [11, §3.5], as follows. Let U_0 be the smooth locus of X. Then, since X is affine, U_0 is a smooth Poisson variety (Remark 1.33) and so is a Poisson manifold. We can stratify U_0 by symplectic leaves, say $U_0 = \bigsqcup_{i \in \mathcal{I}_0} \mathcal{S}_{i,0}$. Now we proceed inductively by setting $X_0 = X$ and defining $X_k = X_{k-1} \setminus U_{k-1}$ for $k \geq 1$. It is known that X_k is an affine Poisson variety by [60, Proposition 15.2.14(i)], so one can, as above, stratify the smooth locus, U_k, of X_k by symplectic leaves, $U_k = \bigsqcup_{i \in \mathcal{I}_k} \mathcal{S}_{i,k}$. Then

$$X = U_0 \sqcup \cdots \sqcup U_t \tag{1.8}$$

for some t and we call

$$X = \bigsqcup_{\substack{i \in \mathcal{I}_k, \\ 0 \leq k \leq t}} \mathcal{S}_{i,k}$$

the *stratification of X by symplectic leaves.*

Example 1.37. By Example 1.27 if V is a symplectic vector space and $G \subset Sp(V)$ is a finite subgroup then V/G is a Poisson variety. Its symplectic leaves have been calculated in [11, Theorem 7.4]. Let $\pi : V \to V/G$ be the orbit map. For a subgroup $H \leq G$ let

$$V_{(H)} = \{v \in V : G_v \text{ is conjugate to } H\}$$

where G_v denotes the stabiliser of v in G. Then

$$V/G = \bigsqcup_{H \leq G} \pi(V_{(H)})$$

is the stratification by symplectic leaves for V/G.

Calculating leaves

We bring together a number of results which are useful for calculating symplectic leaves.

Lemma 1.38. *[55, Proposition 10.5.2] Let $f : M \to N$ be a Poisson map between Poisson manifolds. Let $H \in \mathcal{O}_N^{hol}(N)$. Then $df \circ \Xi_{f^*(H)} = \Xi_H \circ f$ as holomorphic maps $M \to TN$.*

Corollary 1.39. *Let M and N be Poisson manifolds and suppose that M is a Poisson submanifold of N. Then for any $f \in \mathcal{O}^{hol}(N)$ and $m \in M$, $\Xi_f(m) \in T_mM \subseteq T_mN$.*

When considering Hamiltonian vector fields we are only looking at infinitesimal information around each point, a notion made more concrete by the next lemma.

Lemma 1.40. *Let M be a Poisson manifold and suppose that M is an open subvariety of an affine algebraic variety, X. Then for every $m \in M$, $\langle \Xi_f(m) : f \in \mathcal{O}^{hol}(M)\rangle = \langle \Xi_{f|_M}(m) : f \in \mathcal{O}(X)\rangle \subseteq T_mM$.*

Proof. We consider an embedding of X in $\mathbb{C}^l$, for some l, as a closed subvariety. We have $M \subseteq X \subseteq \mathbb{C}^l$, and one can think of M as a complex analytic space in the sense of [70]. Therefore the germs of holomorphic functions at a point $m \in M$, $\mathcal{O}_m^{hol}$, can be identified with $\mathbb{C}\{z_1, \dots, z_l\}/\mathfrak{a}$, where $\mathbb{C}\{z_1, \dots, z_l\}$ is the algebra of power series on $\mathbb{C}^l$ converging on a neighbourhood of m and $\mathfrak{a}$ is the ideal of functions vanishing on X. The obvious inclusion $\mathbb{C}[z_1, \dots, z_m] \to \mathbb{C}\{z_1, \dots, z_l\}$ gives us a homomorphism $\mathbb{C}[z_1, \dots, z_l] \to \mathcal{O}_m^{hol}$ whose kernel is

$$\{f \in \mathbb{C}[z_1, \dots, z_l] : f = 0 \text{ on a complex neighbourhood of } m \text{ in } X\}.$$

This kernel contains I, the defining ideal of X. Therefore we have a homomorphism $\mathcal{O}(X) \to \mathcal{O}_m^{hol}$, and for all $f \in \mathcal{O}(X)$ we denote the image of f under this map by $[f]$.

For an element $h \in \mathcal{O}_m^{hol}$ choose a lift $\tilde{h} \in \mathbb{C}\{z_1, \dots, z_l\}$ of h. Now take the linear term of $\tilde{h}$ and consider the image in $\mathcal{O}(X)$, we denote this element $\overline{h}$. For any $\delta \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_m^{hol}, \mathcal{O}_m^{hol}/\mathbf{m}_m)$ we claim that $\delta(h) = \delta([\overline{h}])$. Let $\phi : \mathbb{C}\{z_1, \dots, z_l\} \to \mathcal{O}_m^{hol}$ be the quotient map, and define a derivation $\mathbb{C}\{z_1, \dots, z_l\} \to \mathcal{O}_m^{hol}/\mathbf{m}_m$ by $\delta \circ \phi$. For any $f \in \mathbb{C}\{z_1, \dots, z_l\}$, $(\delta \circ \phi)(f)$ depends only on the linear part of f. To see this write $f = f_0 + f_1 + f_{\geq 2}$ as a sum of its constant, first order and higher order terms. Then by the Leibniz rule $(\delta \circ \phi)(f_0 + f_1 + f_{\geq 2}) = 0 + (\delta \circ \phi)(f_1) + 0 \in \mathcal{O}_m^{hol}/\mathbf{m}_m$. Now since $\delta(h) = (\delta \circ \phi)(\tilde{h})$ it follows that $\delta(h) = \delta([\overline{h}])$.

Let $f \in \mathcal{O}^{hol}(M)$. Since $M \subseteq \mathbb{C}^l$ we can think of the tangent space $T_m M$ as being a subspace of the ambient space $\mathbb{C}^l$. Therefore we can take the power series expansion of f around m, $[f]$, and view this is as an element of $\mathbb{C}\{z_1, \dots, z_l\}/\mathfrak{a}$. Let $g \in \mathcal{O}^{hol}_m$. Then

$$\Xi_f(m)(g) = \Xi_f(m)([\overline{g}]) = (\Xi_f \cdot \overline{g})(m) = \{f, \overline{g}\}(m) = -\{\overline{g}, f\}(m) = -\Xi_{\overline{g}}(m)([f]).$$

Now repeating this argument we obtain

$$-\Xi_{\overline{g}}(m)([f]) = \Xi_{\overline{[f]}}(m)([\overline{g}]) = \Xi_{\overline{[f]}}(m)(g).$$

Thus $\Xi_f(m) = \Xi_{\overline{[f]}}(m)$. □

Suppose that X is an affine algebraic Poisson variety. The closed Poisson subvarieties of X are important because of the following lemma.

Lemma 1.41. *[11, Lemma 3.5] Suppose that Y is a closed Poisson subvariety of X. Then for every maximal ideal $\mathfrak{m} \in Y$, the symplectic leaf through $\mathfrak{m}$ in X is contained in Y.*

We will have reason to look also at locally closed subvarieties.

Proposition 1.42. *Let X be a Poisson variety and $Y \subseteq X$ a connected open subvariety. Assume that Y is a symplectic manifold, and that the restriction map $\mathcal{O}(X) \to \mathcal{O}^{hol}(Y)$ is Poisson. Then for any $y \in Y$ the symplectic leaf through y in X contains Y.*

Proof. Let $y \in Y$. We prove the proposition by showing that the tangent space to y in Y is spanned by Hamiltonian vector fields $\Xi_{f|_Y}(y)$ for $f \in \mathcal{O}(X)$. We denote the Poisson bracket on $\mathcal{O}(X)$ by $\{-,-\}_X$ and the Poisson bracket on $\mathcal{O}^{hol}(Y)$ induced by the symplectic form by $\{-,-\}_{symp}$. By assumption for all $f, g \in \mathcal{O}(X)$, $\{f, g\}_X|_Y = \{f|_Y, g|_Y\}_{symp}$. The Poisson bracket on $\mathcal{O}(X)$ induces a Poisson bracket on $\mathcal{O}^{hol}(Y)$ which we call $\{-,-\}_Y$, and does so in such a way that restriction $\mathcal{O}(X) \to \mathcal{O}^{hol}(Y)$ is a Poisson map. In other words, for all $f, g \in \mathcal{O}(X)$, $\{f, g\}_X|_Y = \{f|_Y, g|_Y\}_Y$. Therefore $\{f|_Y, g|_Y\}_Y = \{f|_Y, g|_Y\}_{symp}$ for all $f, g \in \mathcal{O}(X)$. By skew symmetry of the Poisson bracket and Lemma 1.40 the Hamiltonian vector fields $\Xi_{f|_Y} = \{f|_Y, -\}_Y$ and $\{f|_Y, -\}_{symp}$ are equal. By Lemma 1.40 and because $\{-,-\}_{symp}$ is the bracket for a symplectic form, $T_y Y$ is spanned by the $\{f|_Y, -\}_{symp}(y)$. □

Symplectic leaves in the algebraic world

The notion of symplectic leaf belongs to differential geometry but we can relate it to certain ideals in the coordinate ring. By a Poisson prime ideal of a Poisson algebra we mean a Poisson ideal which is also a prime ideal.

Theorem 1.43. *Let X be an affine Poisson variety, and $X = \bigsqcup_{i \in \mathcal{I}} \mathcal{S}_i$ its stratification into symplectic leaves. Then*

(i) [11, Lemma 3.5] $\bar{\mathcal{S}}_i = \mathcal{V}(P)$ for all $i \in \mathcal{I}$ where P is Poisson prime ideal of $\mathcal{O}(X)$.

If the index set $\mathcal{I}$ is finite then

(ii) [11, Proposition 3.7] each $\mathcal{S}_i$ is irreducible and locally closed in the Zariski topology and $\mathcal{S}_i = \mathrm{sm}\bar{\mathcal{S}}_i$.

Thus when $\mathcal{I}$ is finite the theorem above gives a one-to-one correspondence between symplectic leaves in X and Poisson prime ideals of $\mathcal{O}(X)$: $\mathcal{S} \longleftrightarrow P$ where $\overline{\mathcal{S}} = \mathcal{V}(P)$.

Recall that the centres of symplectic reflection algebras are Poisson algebras. The corresponding Poisson varieties, X_c, can therefore be stratified into symplectic leaves.

Theorem 1.44. *[11, Theorems 4.2 and 7.8] The stratification of X_c into symplectic leaves is finite. If $\mathfrak{m}, \mathfrak{n}$ are closed points of X_c lying in the same leaf then $H_c/\mathfrak{m}H_c \cong H_c/\mathfrak{n}H_c$.*

This theorem tells us that the representation theory of H_c "is constant across symplectic leaves". Given the corollary below we can view this result as a refinement of Theorem 1.17. It seems clear that it would be desirable to calculate the symplectic leaves of X_c or obtain some information about them. One would hope that given such knowledge then for any closed point $\mathfrak{m}$ in a given leaf one could say something about either $\chi^{-1}(\mathfrak{m})$ or $H_c/\mathfrak{m}H_c$.

In light of Theorem 1.44 it would be interesting to understand these varieties and to calculate their symplectic leaves.

Corollary 1.45. *The smooth locus of X_c is a symplectic leaf.*

Proof. Z_c is a domain, Theorem 1.9, so 0 is a Poisson prime ideal. By Theorem 1.43 (ii), sm $\mathcal{V}(0) =$ sm X_c is a symplectic leaf. $\square$

1.7 Remarks

1. Most of our results on invariant rings were taken from [5], skew group rings are discussed in [60] and in greater generality in [64].

2. Section 1.3 is mostly taken from [26] although some of our results are only implicit in that paper. An excellent introduction to symplectic reflection algebras, which informed our approach, is contained in [8].

3. A more philosophical reason for studying triples for groups generated by symplectic reflections is that the quotient variety only has a symplectic resolution (in the sense of [4]) if the group is generated by symplectic reflections, [77]. In this case the symplectic reflection algebra and the symplectic resolution are expected to be closely related, see [31] and [34] for example.

4. Our version of quantisation is a particular version of the more general definition, although this suffices for our purposes. For a more sophisticated point of view see [47].

5. The contents of Section 1.6 are drawn from a number of sources. The Poisson algebraic geometry is largely based on the exposition in [11], another good reference is [76]. We have based our description of Poisson manifolds on [55] and there are a number of further references therein. Lemma 1.40 uses the relationship between algebraic varieties and manifolds described in [70].

Chapter 2

Poisson geometry and representation theory

We have discussed the relationship between the Poisson geometry of the centres of symplectic reflection algebras and their representation theory. We now look at a slightly different situation in which geometry and representation theory interact, which is in the study of representations of quivers, and, more precisely, representations of deformed preprojective algebras. Much of this material will be used in Chapter 4 to discuss the varieties X_c for wreath products.

2.1 Invariant theory

Recall that we are working over $\mathbb{C}$. Let G be a linearly reductive group acting as morphisms on an affine algebraic variety, X. This induces an action of G on $\mathcal{O}(X)$ in exactly the same way as when G is finite. The ring of invariants, $\mathcal{O}(X)^G$, which is defined as in (1.1), is a finitely generated $\mathbb{C}$-algebra by [5, Theorem 1.6.3]. Therefore one can consider the variety $X//G := \operatorname{Max} \mathcal{O}(X)^G$, and the inclusion $\mathcal{O}(X)^G \to \mathcal{O}(X)$ induces a surjective morphism $\pi : X \to X//G$ which is constant on G-orbits. We call this *the orbit map*. $X//G$ is often referred to as a categorical quotient, which means that it satisfies the following universal property. Let $\phi : X \to Y$ be a morphism of varieties which is constant along the G-orbits in X. Then there exists a unique morphism $\overline{\phi} : X//G \to Y$

such that the diagram below commutes

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\ X//G \arrow[r, "\overline{\phi}"'] & Y. \end{tikzcd}$$

One can think of $X//G$ as parametrising the closed G-orbits in X, in the sense of the following lemma (see [68, Corollary 13.3.1] for a proof).

Lemma 2.1. *For every $\xi \in X//G$ the fibre $\pi^{-1}(\xi)$ is G-stable and contains exactly one closed G-orbit, this is the unique orbit of minimal dimension in $\pi^{-1}(\xi)$.*

A special case where the quotient parametrises all orbits is described the next lemma. For any $x \in X$ the orbit $G \cdot x$ is a locally closed subvariety of X, [68, Theorem 5.4.19]. Therefore it makes sense to talk about the dimension of an orbit.

Corollary 2.2. *Suppose that all orbits in X have the same dimension. Then all orbits are closed in X and for all $\xi \in X//G$ the fibre $\pi^{-1}(\xi)$ consists of a single orbit.*

One consequence of the reductivity of G is that given a G-stable ideal, I, of $\mathcal{O}(X)$ the following sequence is exact

$$0 \to I^G \to \mathcal{O}(X)^G \to (\mathcal{O}(X)/I)^G \to 0.$$

Geometrically this means that for an affine variety X with a closed G-stable subvariety Y one can think of $Y//G$ as a closed subvariety of $X//G$ and the following diagram commutes

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & X \arrow[d] \\ Y//G \arrow[r, hook] & X//G. \end{tikzcd} \tag{2.1}$$

Let $R = \mathcal{O}(X)$ and let $R/I = \mathcal{O}(Y)$. Then the morphism $Y//G \to X//G$ is induced from the algebra homomorphism $R^G \to R^G/I^G$.

One can stratify $X//G$ in a natural way. For $x \in X$ let G_x be the stabiliser of x in G. Let $\mathcal{T}$ be the set of conjugacy classes of subgroups G_x in G. For $\xi \in X//G$ denote by $\mathcal{T}(\xi)$ the conjugacy class of stabilisers belonging to the unique closed orbit in $\pi^{-1}(\xi)$. For each $\tau \in \mathcal{T}$ choose a stabiliser, G_τ, representing the conjugacy class τ, and we write $G_{\tau_1} \leq_c G_{\tau_2}$ if G_{τ_1} is conjugate to a subgroup of G_{τ_2}.

We define a partial order on $\mathcal{T}$ by $\tau_1 \geq \tau_2$ if $G_{\tau_1} \leq_c G_{\tau_2}$. Define $(X//G)_\tau = \{\xi \in X//G : T(\xi) = \tau\}$. We note that if Y is a closed G-stable subvariety of X then one can also stratify $Y//G$ by orbit type. By (2.1) above, $(Y//G)_\tau = (X//G)_\tau \cap Y//G$ for every $\tau \in \mathcal{T}$.

The following is well-known; for a proof see [24, Proposition 2.4].

Proposition 2.3. *The set $\bigcup_{\nu \leq \tau}(X//G)_\nu$ is closed in $X//G$ and $(X//G)_\tau$ is open in $\bigcup_{\nu \leq \tau}(X//G)_\nu$. The stratification $X//G = \bigcup_{\tau \in \mathcal{T}}(X//G)_\tau$ is finite.*

We have more information in the linear case. This Proposition is due to Schwarz, [69, Lemma 5.5], another proof can be found in [24, Proposition 2.5].

Proposition 2.4. *Let X be a finite dimensional vector space on which G acts linearly. Then for each $\tau \in \mathcal{T}$ the stratum $(X//G)_\tau$ is irreducible and its closure is $\bigcup_{\nu \leq \tau}(X//G)_\nu$.*

We can obtain reductive subgroups of G by considering the stabiliser of a point in X.

Lemma 2.5. *[59, Théorème 3] Let $x \in X$ and suppose that the orbit $G \cdot x$ is closed in X. Let G_x be the stabiliser of x in G. Then G_x is a reductive subgroup of G.*

Suppose that G is an affine algebraic group and that H is a reductive subgroup of G and that H acts by morphisms on an affine algebraic variety, Y. Then there is an action of H on the product $G \times Y$ by $h \cdot (g, y) = (gh, h^{-1}y)$ for $h \in H, (g, y) \in G \times Y$; note that this an action on the right. In this situation one form the quotient $(G \times Y)//H$, which is called the *associated bundle of $Y//H$.*

Proposition 2.6. *All orbits for the action of H on $G \times Y$ have the same dimension and are closed. Furthermore, if Y is smooth then $(G \times Y)//H$ is smooth.*

Proof. Let $(g, y) \in G \times Y$. The morphism $H \to G \times Y$; $h \mapsto h \cdot (g, y)$ is injective since $h \cdot (g, y) = (g, y)$ implies that $h = 1$. Therefore $H \cong H \cdot (g, y)$ as algebraic varieties. In particular, all orbits have dimension equal to $\dim H$ and so all orbits are closed in $G \times Y$ by Corollary 2.2. The statement concerning smoothness is proved in [71, II. Proposition 4.22]. $\square$

The proposition and Corollary 2.2 imply that one can identify $(G \times Y)//H$ with the space of H-orbits in $G \times Y$. We consider the projection $p : (G \times Y)//H \to G/H; [g, y] \mapsto gH$, then the fibres are all isomorphic to Y. Associated bundles are locally trivial in the étale topology, but in general not for the Zariski topology, see [71, II. page 195]. Thus if Y is a finite dimensional vector space then $(G \times Y)//H$ is a holomorphic vector bundle over G/H.

Lemma 2.7. *Let Y be a finite dimensional vector space. Let $(1,0) \in G \times Y$ and let $[1,0]$ denote its image in $(G \times Y)//H$. Then*

$$T_{[1,0]}(G \times Y)//H \cong (T_1 G \oplus Y)/T_1 H,$$

where $T_1 H \hookrightarrow T_1 G \oplus T_0 Y$ via the differential of the map $H \to G \times Y; h \mapsto h \cdot (1,0)$ at $1 \in H$.

Proof. This is clear because H embeds in $G \times Y$ as $H \times 0$ so that $T_{[1,0]}(G \times Y)//H \cong (T_1 G/T_1 H \oplus T_0 Y)$. Then one notes that Y is a vector space so that $T_0 Y \cong Y$. $\square$

We give three examples of quotients by reductive algebraic groups which will appear again Chapter 4 in the context of the varieties X_c.

Example 2.8 (Rank one matrices). Let $\mathrm{Mat}_n(\mathbb{C})$ be the variety of $n \times n$ complex matrices, which we can think of as the endomorphism algebra of $\mathbb{C}^n$. It is an n^2 dimensional affine space. Let R_l be the subvariety of matrices whose rank is less than or equal to l. Thus R_l is the set of matrices whose $(l+1) \times (l+1)$ minors are zero and so this is a closed subvariety. The variety R_l is irreducible and has dimension $2nl - l^2$, [12, Proposition 1.1]. Let $\mathbb{C}[x_{ij}]_{1 \le i,j \le n}$ be the coordinate ring of $\mathrm{Mat}_n(\mathbb{C})$, then the defining ideal of R_l is the prime ideal, I_l, of $\mathbb{C}[x_{ij}]_{1 \le i,j \le n}$ generated by the generic $(l+1) \times (l+1)$ minors, [12, Theorem 2.10]. We describe R_1 as a categorical quotient.

Let $V = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$. We identify $\mathbb{C}^n$ with column vectors and $(\mathbb{C}^n)^*$ with row vectors (this is the trace pairing described in the next example). There is an action of $\mathbb{C}^\times$ on V by $\lambda \cdot (c,r) = (\lambda c, \lambda^{-1} r)$. An orbit, $\mathbb{C}^\times \cdot (c,r)$, is closed if and only if $c = r = 0$ or c and r are both nonzero. We choose a basis of V so that its coordinate ring is $\mathbb{C}[c_i, r_j : 1 \le i,j \le n]$. The ring of invariants, $\mathcal{O}(V)^{\mathbb{C}^\times}$, is equal to $\mathbb{C}[c_i r_j : 1 \le i,j \le n]$. Let $U = \{(c,r) \in V : c_1 r_1 + \cdots + c_n r_n \neq 0\}$ so that $(c,r) \in U$ implies that c and r are both nonzero. Clearly U is an open subvariety of V which is $\mathbb{C}^\times$-stable and the orbit of any element in U is closed in V. It follows from (2.1) that the image of U under the orbit map is an open subvariety of $V//\mathbb{C}^\times$ This implies that $\mathrm{Dim}\, V//\mathbb{C}^\times = \dim V - \dim \mathbb{C}^\times = 2n - 1$ by [75, (7) Page 39].

We have a morphism

$$t : V \to R_1;\ (c,r) \mapsto c \otimes r \tag{2.2}$$

which has corresponding comorphism $t^\sharp : \mathbb{C}[x_{ij}]/I_1 \to \mathbb{C}[c_i, r_j]; \overline{x_{ij}} \mapsto c_i r_j$. Clearly t is constant on the $\mathbb{C}^\times$ orbits of V so by the universal property of categorical quotients there exists $\bar{t} : V//\mathbb{C}^\times \to R_1$

making the following diagram commute

$$\begin{tikzcd} V \arrow[d] \arrow[dr, "t"] & \\ V//\mathbb{C}^\times \arrow[r, "\bar{t}"'] & R_1. \end{tikzcd} \tag{2.3}$$

We have

$$\bar{t}^\sharp : \mathbb{C}[x_{ij}]/I_1 \to \mathbb{C}[c_i r_j];\ \overline{x_{ij}} \mapsto c_i r_j, \tag{2.4}$$

this is a surjective map between two prime rings whose Krull dimensions are equal to $2n-1$. Therefore $\bar{t}^\sharp$ is an isomorphism by comparison of Krull dimension, see [25] for example.

Example 2.9 (Calogero-Moser space). We recall some elements from Notation 1.22 from Chapter 1. Let L be a two dimensional symplectic vector space with symplectic form, ω_L. Let $x, y \in L$ be a symplectic basis. Let Γ be a finite subgroup of $\mathrm{SL}(L)$. Let R be the regular representation of Γ, and let $S_0, \ldots, S_k$ (recall that $k+1$ is the number of conjugacy classes in Γ) be the distinct simple representations of Γ which have dimensions $\delta_0, \ldots, \delta_k$ respectively. We let $\delta \in \mathbb{N}^{k+1}$ be the vector with ith entry δ_i. We shall assume once and for all that S_0 is the trivial representation, so that $\delta_0 = 1$.

As a Γ-module, R^n decomposes into a sum of irreducible representations:

$$R^n = \bigoplus_{i=0}^{k} S_i^{\oplus n\delta_i} = \sum_{i=0}^{k} S_i \otimes \mathbb{C}^{n\delta_i}. \tag{2.5}$$

It is clear from the above that one can identify the group $\mathrm{Aut}_\Gamma(R^n)$ with $\hat{G}(n\delta) = \prod_{i=0}^{k} \mathrm{GL}(n\delta_i, \mathbb{C})$.

Denote by $e_\Gamma \in \mathrm{End}_\Gamma(R)$ the projector onto the trivial representation. Let $\underline{c} : \Gamma \setminus \{1\} \to \mathbb{C}$ be a class function. Let $\mathbf{c} \in \mathrm{End}_\Gamma(R)$ denote multiplication by the central element $\sum_{\Gamma\setminus\{1\}} \underline{c}(\gamma)\gamma$. Let $\mathcal{O}$ be the $\mathrm{GL}(n, \mathbb{C})$-conjugacy class formed by all $n \times n$-matrices of the form $P - \mathrm{Id}$, where P is a semisimple rank one matrix such that $\mathrm{tr}(P) = \mathrm{tr}(\mathrm{Id}) = n$. Define

$$Cal_c = \{\nabla \in \mathrm{Hom}_\Gamma(L, \mathrm{End}_\mathbb{C}(R^n)) : [\nabla(x), \nabla(y)] \in \frac{1}{2} c_1 |\Gamma| \mathcal{O} \otimes e_\Gamma + \mathrm{Id} \otimes \mathbf{c}\},$$

where $c_1 \in \mathbb{C}$ and $\frac{1}{2} c_1 |\Gamma| \mathcal{O} \otimes e_\Gamma + \mathrm{Id} \otimes \mathbf{c} \subseteq \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}_\Gamma(R) = \mathrm{End}_\Gamma(\mathbb{C}^n \otimes R) = \mathrm{End}_\Gamma(R^n)$, and where the suffix c on Cal_c stands for $c = (c_1, \underline{c})$.

There is an action of $\hat{\mathrm{G}}(n\delta) = \mathrm{Aut}_\Gamma(R^n)$ by basechange: for all $g \in \mathrm{Aut}_\Gamma(R^n), \nabla \in Cal_c$ and $x \in L$ we have $(g \circ \nabla)(x) = g\nabla(x)g^{-1}$. The action factors through $\mathbb{C}^\times$ so that $\mathrm{G}(n\delta) := \hat{\mathrm{G}}(n\delta)//\mathbb{C}^\times$

acts on Cal_c. The space $Cal_c/\!/\,\mathrm{G}(n\delta)$ is called *Calogero-Moser Space for* Γ_n. There is a discussion of this space in [26, §11].

The relationship with the varieties, X_c, is as follows. Recall the wreath product, Γ_n, acting on $(\mathbb{C}^2)^{\oplus n}$. The parameter c, which is a class function on symplectic reflections of Γ_n, can be identified with a pair, $(c_1, \underline{c})$, as in Section 1.4. Here $c_1 \in \mathbb{C}$ and $\underline{c}$ is a class function $\Gamma \setminus \{1\} \to \mathbb{C}$. In other words, the parameter c for symplectic reflection algebras of wreath products is the same as the c described in the paragraphs above. The method of proof of Theorem 1.23 is to establish the following.

Theorem 2.10. *[26, Proposition 11.11 and Theorem 11.16] For generic values of c, $Cal_c/\!/\,\mathrm{G}(n\delta)$ is smooth and is isomorphic to X_c.*

Example 2.11 (Representations of quivers). Let Q be a quiver with vertex set I, set of arrows A, and let $h, t : A \to I$ denote the head and tail functions respectively. If $\alpha \in \mathbb{N}^I$, the space of representations of Q of dimension vector α is

$$\mathrm{Rep}(Q, \alpha) = \bigoplus_{a \in A} \mathrm{Mat}(\alpha_{h(a)} \times \alpha_{t(a)}, \mathbb{C}).$$

The group

$$\mathrm{G}(\alpha) = (\prod_{i \in I} \mathrm{GL}(\alpha_i, \mathbb{C}))/\mathbb{C}^\times$$

acts by conjugation on $\mathrm{Rep}(Q, \alpha)$. It is clear that $\mathrm{Rep}(Q, \alpha)$ parametrises representations of the path algebra $\mathbb{C}Q$ of dimension vector α, and that the $\mathrm{G}(\alpha)$-orbits are the isomorphism classes of representations. Therefore we shall often think of points of $\mathrm{Rep}(Q, \alpha)$ as modules for the corresponding path algebra and orbits as isomorphism classes. We can form the quotient $\mathrm{Rep}(Q, \alpha)/\!/\,\mathrm{G}(\alpha)$, which parametrises the closed orbits in $\mathrm{Rep}(Q, \alpha)$.

Proposition 2.12. *[3, Section 12] The closed orbits in* $\mathrm{Rep}(Q, \alpha)$ *are precisely the isomorphism classes of semisimple representations of* $\mathbb{C}Q$.

Therefore the points in $\mathrm{Rep}(Q, \alpha)/\!/\,\mathrm{G}(\alpha)$ are the isomorphism classes of semisimple representations of $\mathbb{C}Q$. We can give an elegant description of the stratification by orbit type for these quotients in terms of modules of the path algebra of Q.

If M is a semisimple $\mathbb{C}Q$-module then we can decompose it into its simple components $M = M_1^{\oplus k_1} \oplus \cdots \oplus M_r^{\oplus k_r}$ where the M_t are non-isomorphic simples. If $\beta^{(t)}$ is the dimension vector of M_t, then we say M has representation type

$$(k_1, \beta^{(1)}; \ldots ; k_r, \beta^{(r)}).$$

This is, of course, only defined up to permutation of the isotypic components. Let $\tau = (k_1, \beta^{(1)}; \ldots ; k_r, \beta^{(r)})$ and let $\mathcal{Q}_\tau$ be the subset of $\mathrm{Rep}(Q, \alpha)// \mathrm{G}(\alpha)$ with representation type τ. We shall refer to the stratification $\mathrm{Rep}(Q, \alpha)// \mathrm{G}(\alpha) = \bigsqcup_{\tau; \mathcal{Q}_\tau \neq \emptyset} \mathcal{Q}_\tau$ as *the stratification by representation type for* $\mathrm{Rep}(Q, \alpha)// \mathrm{G}(\alpha)$.

Theorem 2.13. *[53, Theorem 2] The stratification of* $\mathrm{Rep}(Q, \alpha)// \mathrm{G}(\alpha)$ *by representation type is equal to the stratification by orbit type.*

2.2 Moment maps and reduction

We now look at the interplay between invariant theory and Poisson geometry. We show how to perform reduction for symplectic varieties to produce Poisson varieties and give as an example reduction for representations of quivers.

Let M be a symplectic variety as in Section 1.6. Let ω be the 2-form on M and let $\{-,-\}$ be the Poisson bracket induced from ω. A vector field X is *symplectic* if it preserves ω, that is, $L_X \omega = 0$, where L_X is the Lie derivative with respect to X. Let $\mathrm{Symp}(M)$ denote the Lie subalgebra of symplectic vector fields on M.

Proposition 2.14. *[15, Proposition 1.2.5]* Ξ_f *is a symplectic vector field for all* $f \in \mathcal{O}(M)$, *and the map* $f \mapsto \Xi_f$ *defines a Lie algebra homomorphism* $(\mathcal{O}(M), \{-,-\}) \to (\mathit{Symp}(M), [-,-])$.

Suppose that a reductive algebraic group G acts on M preserving ω. That is, if $\phi_g : M \to M$ denotes the action of $g \in G$ then, for all $m \in M$, $d\phi_g : T_m M \to T_{m^g} M$ intertwines the symplectic forms. Let $\mathfrak{g}$ be the Lie algebra of G.

Recall the infinitesimal action of $\mathfrak{g}$ on $\mathcal{O}(M)$: for each $x \in \mathfrak{g}$ and $f \in \mathcal{O}(M)$, $x_M(f) := \frac{d}{dt}(exp(tx) \circ f)|_{t=0}$. The operator x_M is a derivation of $\mathcal{O}(M)$ and so defines a map $\mathfrak{g} \to \mathrm{Vect}(M)$. This map is a Lie algebra homomorphism, [55, Proposition 9.3.6], and one can easily check that its image is contained in the space of symplectic vector fields.

The G-action is said to be *Hamiltonian* if there exists a Lie algebra homomorphism $H : \mathfrak{g} \to (\mathcal{O}(M), \{-,-\})$ such that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \arrow[d, "H"'] \arrow[dr] & \\ \mathcal{O}(M) \arrow[r] & \mathrm{Symp}(M). \end{tikzcd} \tag{2.6}$$

Suppose that the G-action is Hamiltonian and let $H_x = H(x)$ for all $x \in \mathfrak{g}$. The *moment map* $\mu : M \to \mathfrak{g}^*$ is the morphism of algebraic varieties defined by $\mu(m)(x) = H_x(m)$ for all $m \in M$ and $x \in \mathfrak{g}$.

Example 2.15. Recall that the coordinate ring, $\mathcal{O}(\mathfrak{g}^*) = S\mathfrak{g}$, of $\mathfrak{g}^*$ has a canonical Poisson bracket induced from the Lie bracket on $\mathfrak{g}$. One can calculate the infinitesimal action of $\mathfrak{g}$ on $S\mathfrak{g}$ (see [55], Examples (c), Page 272): for $x \in \mathfrak{g}$ and $f \in \mathfrak{g} \subset S\mathfrak{g}$, $x_{\mathfrak{g}^*}(f) = [x, f]$. Clearly the action of G is Hamiltonian, with $H_x = x$ for all $x \in \mathfrak{g}$, and the corresponding moment map is $\mu = \mathrm{Id}_{\mathfrak{g}^*}$.

Let $\mathcal{L}$ be a closed orbit (under the coadjoint action) in $\mathfrak{g}^*$. Then $\mathcal{L}$ is a symplectic variety and its Poisson bracket is simply the restriction of the bracket of $S\mathfrak{g}$ ([15, Proposition 1.3.21]). Now if we consider $\mathcal{L}$ as a closed subvariety of $\mathfrak{g}^*$ then its defining ideal, I, is a Poisson ideal of $S\mathfrak{g}$. For any $x \in \mathfrak{g}$, one calculates that $x_{\mathcal{L}}$ is the derivation of $S\mathfrak{g}/I$ given by $x_{\mathcal{L}}(f + I) = [x, f] + I$ for all $f \in \mathfrak{g} \subset S\mathfrak{g}$. This makes sense because I is Poisson. Therefore the action of G on $\mathcal{L}$ is Hamiltonian with $H_x = x + I$. The corresponding moment map is the inclusion $\mathcal{L} \hookrightarrow \mathfrak{g}^*$.

Remark 2.16. We note a feature of the setup (2.6) that will appear later in Chapter 4. Suppose that G contains a normal subgroup K which acts trivially on M. Let $\mathfrak{k} = \mathrm{Lie}\, K$. Then the infinitesimal action of $\mathfrak{k}$ on $\mathcal{O}(M)$ is simply the zero map. Thus in order for diagram (2.6) to commute we must have that the restriction of H to $\mathfrak{k}$ is contained in the set $\{f \in \mathcal{O}(M) : \{f, g\} = 0 \text{ for all } g \in \mathcal{O}(M)\}$ which is called the subalgebra of *Casimir* functions. Now if we evaluate $\mu(m)$ at an element $A \in \mathfrak{k} \subseteq \mathfrak{g}$ we see that this equals $H_A(m)$, which need not be zero. Therefore there is no reason why the image of the moment map should be contained in $\mathrm{Lie}(G/K)^* \subseteq \mathrm{Lie}\, G^*$.

The moment map has the following properties.

Proposition 2.17. *[15, Lemma 1.4.2] The comorphism*

$$\mu^\sharp : \mathbb{C}[\mathfrak{g}] \to \mathcal{O}(M)$$

induced by μ is a Poisson homomorphism, and if G is connected then μ is G-equivariant (relative to the coadjoint action on $\mathfrak{g}^*$).

In general it is interesting to know whether a group action on a symplectic variety is Hamiltonian. In the linear case this is known.

Theorem 2.18. *[15, Proposition 1.4.6] Let V be a symplectic vector space with symplectic form ω and let G be a reductive algebraic subgroup of $Sp(V)$.. Then the action is Hamiltonian with $H_A(v) = \frac{1}{2}\omega(A \cdot v, v)$ for all $A \in$ Lie G and $v \in V$. The corresponding moment map is G-equivariant.*

We can construct symplectic varieties with Hamiltonian actions from old ones by forming products.

Lemma 2.19. *Let M_1, M_2 be two symplectic varieties. Suppose that a reductive algebraic group G acts on each variety in a Hamiltonian fashion. Then the product, $M_1 \times M_2$, is a symplectic variety and the diagonal action of G is Hamiltonian. Furthermore, if μ_1, μ_2 are moment maps for the G-action on M_1 and M_2, respectively, then $\mu_1 + \mu_2 : M_1 \times M_2 \longrightarrow \mathfrak{g}^*; (m_1, m_2) \mapsto \mu_1(m_1) + \mu_2(m_2)$ is a moment map for the action of G on the product.*

Proof. It is clear that $M_1 \times M_2$ is symplectic. Denote the Hamiltonian maps for M_1 and M_2 by H^1 and H^2 respectively. It follows from the product rule that $H_x = H_x^1 \otimes 1 + 1 \otimes H_x^2 \in \mathcal{O}(M_1 \times M_2) = \mathcal{O}(M_1) \otimes \mathcal{O}(M_2)$ is a Hamiltonian for the G-action on $M_1 \times M_2$. The corresponding moment map is given by $\mu(m_1, m_2)(x) = H_x(m_1, m_2) = H_x^1(m_1) + H_x^2(m_2) = \mu_1(m_1)(x) + \mu_2(m_2)(x)$ for all $(m_1, m_2) \in M_1 \times M_2, x \in \mathfrak{g}$. $\square$

Reduction

Many of the varieties which we shall study arise as quotient spaces of fibres of a moment map and we shall see now how these spaces carry a Poisson structure. Let (M, ω) be a symplectic variety and let the connected reductive algebraic group G act on M preserving ω. Suppose this action is Hamiltonian and let μ denote the corresponding moment map. Let $\mathcal{L}$ be a closed orbit under the coadjoint action of G on $\mathfrak{g}^*$, with defining G-stable ideal $J \lhd \mathbb{C}[\mathfrak{g}]$. Then $\mu^{-1}(\mathcal{L})$ is a G-stable closed subset of M by Proposition 2.17. Let $I \lhd \mathcal{O}(M)$ be the defining G-stable radical ideal of $\mu^{-1}(\mathcal{L})$.

We consider the quotient variety $M_{\mathcal{L}} := \mu^{-1}(\mathcal{L})//G$ - this space is called a *Marsden-Weinstein reduction.* Let $\{-,-\}$ be the bracket on $\mathcal{O}(M)$. Since G is reductive $\left(\mathcal{O}(M)/I\right)^G \cong \mathcal{O}(M)^G/I^G$ so one can define a bracket on $\mathcal{O}(M_{\mathcal{L}}) = \left(\mathcal{O}(M)/I\right)^G$ by defining one on $\mathcal{O}(M)^G/I^G$. Define a bracket, $\{-,-\}'$, on $\mathcal{O}(M)^G/I^G$ by $\{f+I^G, g+I^G\}' = \{f,g\}+I^G$ for all $f,g \in \mathcal{O}(M)^G$.

Proposition 2.20. *The bracket $\{-,-\}'$ is well-defined and is a Poisson bracket on $\mathcal{O}(M)^G/I^G$.*

Proof. It is clear that $\{-,-\}'$ will be a Poisson bracket as long as it is well defined. To see this we note first that $\{f,g\} \in \mathcal{O}(M)^G$ for all $f,g \in \mathcal{O}(M)^G$. It remains to show that I^G is a Poisson ideal of $\mathcal{O}(M)^G$.

For all $x \in \mathfrak{g}$, $\mu^\sharp(x) = x \circ \mu$ and if we evaluate this at an element of $m \in M$ we see that $(x \circ \mu)(m) = x(H_-(m)) = H_x(m)$. In short, $\mu^\sharp(x) = H_x$ and it follows that $I' := \mathcal{O}(M)\mu^\sharp(J)$ is generated by polynomials in the H_x.

Now, if $f \in \mathcal{O}(M)^G$ then $h \cdot f = f$ for all $h \in G$ and therefore $x_M f = 0$ for all $x \in \mathfrak{g}$. Thus $\{H_x, f\} = x_M f = 0$ and, by the product rule, $\{f,i\} \in I'$ for all $i \in I'$. Therefore I'^G is a Poisson ideal of $\mathcal{O}(M)^G$. Finally, $I = \sqrt{I'}$ implies that $I^G = \sqrt{I'^G}$, and so I^G is a Poisson ideal of $\mathcal{O}(M)^G$ by [23, 3.3.2]. $\square$

Therefore Marsden-Weinstein reductions are Poisson varieties. To compare the Poisson brackets of two reductions it suffices to compare the Poisson structures of the original symplectic varieties.

Lemma 2.21. *Suppose that we have two symplectic varieties, V and W, both of which have the Hamiltonian action of a connected reductive algebraic group, G. Denote moment maps for these actions by μ and ν respectively. Suppose there is a G-equivariant Poisson map $\phi : V \to W$ which restricts to an isomorphism $\mu^{-1}(\mathcal{L}) \to \nu^{-1}(\mathcal{O})$ for some closed coadjoint orbits, $\mathcal{L}, \mathcal{O} \subset \mathfrak{g}^*$. Then ϕ induces a Poisson isomorphism of the reductions $\mu^{-1}(\mathcal{L})//G \cong \nu^{-1}(\mathcal{O})//G$.*

Proof. Let I and J be the defining ideals of $\mu^{-1}(\mathcal{L})$ and $\nu^{-1}(\mathcal{O})$ respectively. Then the comorphism $\phi^\sharp : \mathcal{O}(W) \to \mathcal{O}(V)$ induces an isomorphism $\mathcal{O}(W)/J \to \mathcal{O}(V)/I; f+J \mapsto \phi^\sharp(f)+I$. Since ϕ is equivariant we get an isomorphism $\mathcal{O}(X)^G/J^G \to \mathcal{O}(V)^G/I^G$ in a similar way. Now since $\phi^\sharp$ is Poisson this last isomorphism is Poisson by Proposition 2.20. $\square$

Example 2.22 (Rank one matrices). Recall the notation from Example 2.8. Consider the symplectic vector space $V = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$ with form given by $\omega((c,r),(c',r')) = -rc' + r'c$. The action of $\mathbb{C}^\times$ preserves ω so we have a moment map $\mu : V \to \mathbb{C}^*, (c, r) \mapsto (\lambda \mapsto (rc)\lambda)$, see Theorem 2.18. We identify $\mathbb{C}^*$ with $\mathbb{C}$ via the trace pairing. Since $\mathbb{C}^\times$ is abelian the coadjoint action is trivial so the orbits are just points in $\mathbb{C}$. For any $k \in \mathbb{C}$, $\mu^{-1}(k)$ is the set of points vanishing at $c_1r_1 + \cdots + c_nr_n - k$. The Marsden-Weinstein reduction $\mu^{-1}(k)//\mathbb{C}^\times$ is a Poisson variety. Given the calculation of $\mathcal{O}(V)^{\mathbb{C}^\times}$ from Example 2.8, one sees that

$$\mathcal{O}(\mu^{-1}(k)//\mathbb{C}^\times) = \mathbb{C}[c_ir_j : 1 \leq i, j \leq n]/\sqrt{(c_1r_1 + \cdots + c_nr_n - k)}$$

and the Poisson form is induced by the symplectic form on V, Proposition 2.20. In fact, we can explicitly calculate the bracket on $\mathcal{O}(V//\mathbb{C}^\times) = \mathbb{C}[c_ir_j : 1 \leq i, j \leq n]$ as $\{c_ir_j, c_kr_l\} = -\delta_{jk}c_ir_l + \delta_{il}c_kr_j$, and then it is easy to verify that $(c_1r_1 + \cdots + c_nr_n - k)$ is a Poisson ideal and therefore so is its radical.

On the other hand $\mathrm{Mat}_n(\mathbb{C})$ is the Lie algebra of $\mathrm{GL}(n, \mathbb{C})$ so can be identified with with its dual via the trace pairing and in this way becomes a Poisson variety. The Poisson bracket on $\mathbb{C}[x_{ij} : 1 \leq i, j \leq n]$ is $\{x_{ij}, x_{kl}\} = -\delta_{jk}x_{il} + \delta_{il}x_{kj}$. By [60, Proposition 15.2.14], I_1 is a Poisson ideal, and a direct calculation shows that $(x_{11} + \cdots + x_{nn} - k)$ is also a Poisson ideal. It follows from (2.4) and the description of the Poisson brackets that there is a Poisson isomorphism

$$\mathbb{C}[x_{ij}]/I_1 \cong \mathbb{C}[c_ir_j].$$

Furthermore, this descends to a Poisson isomorphism of Poisson subvarieties, that is,

$$\mathbb{C}[x_{ij}]/I_1 + \sqrt{(x_{11} + \cdots + x_{nn} - k)} \to \mathbb{C}[c_ir_j]/\sqrt{(c_1r_1 + \cdots + c_nr_n - k)},$$

is a Poisson isomorphism. Let $U_k = \mu^{-1}(k)//\mathbb{C}^\times$ and let $\mathcal{L}_k$ be the variety of matrices of rank one whose trace equals k. We have shown that both U_k and $\mathcal{L}_k$ are Poisson varieties and that they are Poisson isomorphic.

For later use we note an additional property of the isomorphism $U_k \cong \mathcal{L}_k$. Let $G = \mathrm{GL}(n, \mathbb{C})$. There is a canonical action of G on V and one on $\mathrm{Mat}_n(\mathbb{C})$ by conjugation. We can embed $\mathbb{C}^\times$ in the centre of G as the scalar elements. Therefore G acts on both $V//\mathbb{C}^\times$ and R_1. Since the map t from (2.3) is G-equivariant, the map $\bar{t}$ is also. The trace of a matrix is invariant under conjugation so that $\mathcal{L}_k$ is G-stable. Thus the isomorphism

$$U_k \cong \mathcal{L}_k \tag{2.7}$$

is G-equivariant.

Example 2.23 (Calogero-Moser space). Recall the notation from Notation 1.22, and the definition of Calogero-Moser space. The symplectic form, ω_L, defines a symplectic form on L^* via the isomorphism $L \cong L^*$, which we also denote by ω_L. The vector space $L^* \otimes \operatorname{End}_{\mathbb{C}}(R^n)$ is symplectic with form $\omega_L \otimes tr$, where tr is the symmetric bilinear form $tr(\phi, \psi) = tr_{R^n}(\phi\psi)$ for $\phi, \psi \in \operatorname{End}_{\mathbb{C}}(R^n)$. The form $\omega_L \otimes tr$ is Γ-invariant so the subspace of Γ-invariants, $(L^* \otimes \operatorname{End}_{\mathbb{C}}(R^n))^{\Gamma}$, is also symplectic. The vector space $\operatorname{Hom}_{\Gamma}(L, \operatorname{End}_{\mathbb{C}}(R^n))$ is isomorphic to $(L^* \otimes \operatorname{End}_{\mathbb{C}}(R^n))^{\Gamma}$ meaning that we can define a symplectic form on the former, and the action of $\mathrm{G}(n\delta)$ preserves this form. The Lie algebra of $\hat{\mathrm{G}}(n\delta)$, $\operatorname{Lie}\hat{\mathrm{G}}(n\delta)$, is $\operatorname{End}_{\Gamma}(R^n)$, and there is an isomorphism (which we refer to as the trace pairing)

$$\operatorname{End}_{\Gamma}(R^n) \to (\operatorname{End}_{\Gamma}(R^n))^*;\ A \mapsto tr(A, -).$$

Then $\operatorname{Lie}\mathrm{G}(n\delta) = \operatorname{End}_{\Gamma}(R^n)/\mathbb{C}$, where we identify $\mathbb{C}$ in the latter with $\{\lambda \cdot \mathrm{Id}_{R^n} : \lambda \in \mathbb{C}\}$, and the trace pairing restricts to an isomorphism

$$\operatorname{End}^0_{\Gamma}(R^n) = \{A \in \operatorname{End}_{\Gamma}(R^n) : tr_{R^n}A = 0\} \cong (\operatorname{Lie}\mathrm{G}(n\delta))^*.$$

We consider the map

$$\operatorname{Hom}_{\Gamma}(L, \operatorname{End}_{\mathbb{C}}(R^n)) \to \operatorname{End}^0_{\Gamma}(R^n);\ \nabla \mapsto [\nabla(x), \nabla(y)].$$

We need to check that $[\nabla(x), \nabla(y)] \in \operatorname{End}_{\Gamma}(R^n)$. Let $x^*, y^* \in L^*$ be the dual basis to x, y respectively so that for any $\nabla \in \operatorname{Hom}_{\Gamma}(L, \operatorname{End}_{\mathbb{C}}(R^n))$ we can write ∇ as a Γ-invariant element $x^* \otimes \phi + y^* \otimes \psi$ where $\phi, \psi \in \operatorname{End}_{\mathbb{C}}(R^n)$. Let $\gamma \in \Gamma$. Since $Sp(L) = \mathrm{SL}(L)$ we have $(x^*)^\gamma = ax^* + cy^*, (y^*)^\gamma = bx^* + dy^*$ where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. We have

$$\begin{aligned}
&\phi\psi - \psi\phi = [\nabla(x), \nabla(y)] = [\nabla^\gamma(x), \nabla^\gamma(y)] \\
&= [(ax^* + cy^*)(x) \otimes \phi^\gamma + (bx^* + dy^*)(x) \otimes \psi^\gamma, (ax^* + cy^*)(y) \otimes \phi^\gamma + (bx^* + dy^*)(y) \otimes \psi^\gamma] \\
&= [a\phi^\gamma + b\psi^\gamma, c\phi^\gamma + d\psi^\gamma] = (ad - bc)(\phi^\gamma\psi^\gamma - \psi^\gamma\phi^\gamma) = \phi^\gamma\psi^\gamma - \psi^\gamma\phi^\gamma.
\end{aligned}$$

Thus $[\nabla(x), \nabla(y)]$ is a Γ-invariant element of $\operatorname{End}_{\mathbb{C}}(R^n)$, or, in other words, $[\nabla(x), \nabla(y)] \in \operatorname{End}_{\Gamma}(R^n)$.

Again writing $\nabla = x^* \otimes \phi + y^* \otimes \psi$ we calculate that, for any $A \in \operatorname{End}_{\Gamma}(R^n)/\mathbb{C}$ (we abuse notation by writing A for its coset representative also),

$$tr_{R^n}([\nabla(x), \nabla(y)]A) = tr_{R^n}(\phi\psi A - \psi\phi A) = \tfrac{1}{2}tr_{R^n}(\phi\psi A - \psi\phi A + \phi\psi A - \psi\phi A) \tag{2.8}$$

$$= \tfrac{1}{2}tr_{R^n}((A\phi - \phi A)\psi - (A\psi - \psi A)\phi) = \tfrac{1}{2}(\omega_L \otimes tr)(A \cdot \nabla, \nabla). \tag{2.9}$$

The final equality is true because $A \cdot \nabla = x^* \otimes (A\phi - \phi A) + y^* \otimes (A\psi - \psi A)$. Therefore by Theorem 2.18 the action of $\mathrm{G}(n\delta)$ is Hamiltonian with moment map $\nabla \mapsto [\nabla(x), \nabla(y)]$ and so the space $Cal_c // \mathrm{G}(n\delta)$ is a Marsden-Weinstein reduction and in particular is a Poisson variety. This additional structure is preserved in the isomorphism of Theorem 2.10.

Proposition 2.24. *[26, Theorem 11.16] When it exists, the isomorphism between X_c and $Cal_c//\mathrm{G}(n\delta)$ is Poisson.*

We note now for later use that a straightforward calculation gives another description of the symplectic form on $\mathrm{Hom}_\Gamma(L, \mathrm{End}_{\mathbb{C}}(R^n))$, that is,

$$\omega_L \otimes tr(\nabla, \nabla') = \mathrm{tr}_{R^n}(\nabla(x)\nabla'(y) - \nabla'(x)\nabla(y)). \tag{2.10}$$

Example 2.25 (Representations of deformed preprojective algebras). We perform a reduction procedure for representations of quivers. To begin we need a symplectic variety, but there is no obvious symplectic structure on $\mathrm{Rep}(Q, \alpha)$, so we perform a construction to induce one. Let Q be a quiver, and recall the notation from Example 2.11. Given $\alpha \in \mathbb{N}^I$, $\mathrm{Rep}(Q, \alpha)$ is an affine space. Let $\overline{Q}$ denote the double of Q which is the quiver obtained from Q by adjoining a reverse arrow $a^* : j \to i$ for each arrow $a : i \to j$ in A. We denote the set of arrows in $\overline{Q}$ by $\overline{A}$. Through the trace pairing (see previous example),

$$\mathrm{Rep}(\overline{Q}, \alpha) = \bigoplus_{a \in A} \mathrm{Mat}(\alpha_{h(a)} \times \alpha_{t(a)}, \mathbb{C}) \oplus \bigoplus_{a \in A} \mathrm{Mat}(\alpha_{h(a^*)} \times \alpha_{t(a^*)}, \mathbb{C})$$

is the cotangent bundle of $\mathrm{Rep}(Q, \alpha)$. To keep the notation manageable we let $\underline{B} = (B_a) \in \bigoplus_{a \in A} \mathrm{Mat}(\alpha_{h(a)} \times \alpha_{t(a)}, \mathbb{C})$ and $\underline{B}^* = (B_{a^*}) \in \bigoplus_{a \in A} \mathrm{Mat}(\alpha_{h(a^*)} \times \alpha_{t(a^*)}, \mathbb{C})$. There is a canonical symplectic form on $\mathrm{Rep}(\overline{Q}, \alpha)$ by

$$\omega((\underline{B}, \underline{B}^*), (\underline{C}, \underline{C}^*)) = \sum_{a \in A} -\mathrm{tr}\,(B_{a^*} C_a) + \mathrm{tr}\,(C_{a^*} B_a)$$

for all $(\underline{B}, \underline{B}^*), (\underline{C}, \underline{C}^*) \in \mathrm{Rep}(\overline{Q}, \alpha)$.

The action of $G(\alpha)$ extends to $\mathrm{Rep}(\overline{Q}, \alpha)$ in the obvious way and preserves the symplectic form. By Theorem 2.18 this action is Hamiltonian and one can easily verify (see [16, page 258] for example) that the corresponding moment map is

$$\mu_\alpha : \mathrm{Rep}(\overline{Q}, \alpha) \longrightarrow \mathrm{End}(\alpha)_0, \quad \mu_\alpha(\underline{B}, \underline{B}^*)_i = \sum_{a \in A, h(a) = i} B_a B_{a^*} - \sum_{a \in A, t(a) = i} B_{a^*} B_a$$

where

$$\operatorname{End}(\alpha)_0 = \{(\theta_i) \in \bigoplus_{i\in I} \operatorname{Mat}(\alpha_i \times \alpha_i, \mathbb{C}) : \sum_{i\in I} \operatorname{tr}(\theta_i) = 0\} \cong (\operatorname{Lie} \mathrm{G}(\alpha))^*,$$

and $\operatorname{Lie} \mathrm{G}(\alpha)$ is isomorphic to its dual via the trace pairing.

The coadjoint action of $\mathrm{G}(\alpha)$ on $\operatorname{End}(\alpha)_0$ is by simultaneous conjugation. Given $\lambda \in \mathbb{C}^I$ with $\lambda \cdot \alpha := \sum_{i\in I} \lambda_i \alpha_i = 0$, there is a $\mathrm{G}(\alpha)$-invariant element of $\operatorname{End}(\alpha)_0$ whose ith component is $\lambda_i \mathrm{Id}_{\alpha_i}$ - this element is fixed under the coadjoint action of $\mathrm{G}(\alpha)$ on $(\operatorname{Lie} \mathrm{G}(\alpha))^*$, and so is a coadjoint orbit. The corresponding Marsden-Weinstein reductions are

$$\mathcal{N}(\lambda, \alpha) := \mu_\alpha^{-1}(\lambda) // \mathrm{G}(\alpha).$$

Given that the spaces $\operatorname{Rep}(Q, \alpha) // \mathrm{G}(n\delta)$ described certain isomorphism classes of representations of the path algebra of Q, it is not surprising that the $\mathcal{N}(\lambda, \alpha)$ have a similar interpretation.

Given $\lambda \in \mathbb{C}^I$ define the deformed preprojective algebra

$$\Pi_\lambda = \frac{\mathbb{C}\overline{Q}}{(\sum_{a\in A} [a, a^*] - \sum_{i\in I} \lambda_i e_i)},$$

where $[a, a^*] = aa^* - a^*a$. This algebra is independent of the orientation of Q. Let $\alpha \in \mathbb{N}^I$ be such that $\lambda \cdot \alpha = 0$. The subset $\mu_\alpha^{-1}(\lambda)$ consists of representations of $\mathbb{C}\overline{Q}$ which are annhilated by $(\sum_{a\in A} [a, a^*] - \sum_{i\in I} \lambda_i e_i)$. Thus the representations of Π_λ of dimension α can be $\mathrm{G}(\alpha)$-equivariantly identified with $\mu_\alpha^{-1}(\lambda)$. An analogous result to Proposition 2.12 is true for Π_λ (the proof given in [3, Section 12] works here also), so $\mathcal{N}(\lambda, \alpha)$ classifies the isomorphism classes of semisimple representations of Π_λ of dimension α. It is known if $\lambda \cdot \alpha \neq 0$ then Π_λ has no representations of dimension α, [16, Theorem 1.2].

Any semisimple Π_λ-module M has a representation type as in Example 2.11, and one denotes the subset of $\mathcal{N}(\lambda, \alpha)$ consisting of isomorphism classes with representation type τ by $\mathcal{R}_\tau$. Therefore can can stratify $\mathcal{N}(\lambda, \alpha)$ according to representation type.

By (2.1) one can think $\mathcal{N}(\lambda, \alpha)$ as a closed subset of $\operatorname{Rep}(\overline{Q}, \alpha) // \mathrm{G}(\alpha)$ and, in the notation of Section 2.1, for any $\sigma \in \mathcal{T}$,

$$(\mathcal{N}(\lambda, \alpha))_\sigma = \mathcal{N}(\lambda, \alpha) \cap (\operatorname{Rep}(\overline{Q}, \alpha) // \mathrm{G}(\alpha))_\sigma.$$

Also, for any representation type τ, using the notation of the paragraph preceding Theorem 2.13

$$\mathcal{R}_\tau = \mathcal{N}(\lambda, \alpha) \cap (\overline{Q})_\tau.$$

The next lemma now follows easily from Theorem 2.13.

Lemma 2.26. *The stratifications of $\mathcal{N}(\lambda, \alpha)$ by representation type and by orbit type are equal.*

2.3 Roots and simple dimension vectors of deformed preprojective algebras

We examine some of the properties of the reductions $\mathcal{N}(\lambda, \alpha)$. There are a number of beautiful theorems relating the root combinatorics of quivers to geometric properties of these varieties and their representation type strata. It is these connections which we seek to exploit in the context of symplectic reflection algebras.

Let Q, A, I, h, t be as in Example 2.11. We begin with some definitions. Elements of $\mathbb{Z}^I$ are *vectors* and we write ϵ_i for the coordinate vector at vertex i. We say that a vector α has *connected support* if the quiver with vertices $\{i \in I : \alpha_i \neq 0\}$ and arrows $\{a \in A : \alpha_{h(a)}, \alpha_{t(a)} \neq 0\}$ is connected. We partially order $\mathbb{Z}^I$ via $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for all i, and we write $\alpha > \beta$ to mean that $\alpha \geq \beta$ and $\alpha \neq \beta$. A vector α is *positive* if $\alpha > 0$, and *negative* if $\alpha < 0$. For any subset $X \subseteq \mathbb{Z}^I$ we denote the positive and negative vectors in X by X^+ and X^- respectively.

The *Ringel form* on $\mathbb{Z}^I$ is defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in I} \alpha_i \beta_i - \sum_{a \in A} \alpha_{t(a)} \beta_{h(a)},$$

and let $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ be its symmetrisation. Define

$$p(\alpha) = 1 + \sum_{a \in A} \alpha_{t(a)} \alpha_{h(a)} - \alpha \cdot \alpha = 1 - \frac{1}{2}(\alpha, \alpha).$$

The *fundamental region*, $\mathcal{F}$, is the set $0 \neq \alpha \in \mathbb{N}^I$ with connected support and with $(\alpha, \epsilon_i) \leq 0$ for every vertex i. If i is a loopfree vertex (so $p(\epsilon_i) = 0$), there is a reflection $s_i : \mathbb{Z}^I \to \mathbb{Z}^I$ defined by

$$s_i(\alpha) = \alpha - (\alpha, \epsilon_i)\epsilon_i.$$

The *real roots*, *Re*, are the elements of $\mathbb{Z}^I$ which can be obtained from the coordinate vector at a loopfree vertex by applying a sequence of reflections at loopfree vertices. The *imaginary roots*, *Im*, are the elements of $\mathbb{Z}^I$ which can be obtained from $\mathcal{F} \cup -\mathcal{F}$ by a sequence of reflections at loopfree vertices. For a quiver without loops it is easy to see that $\alpha \in Re$ implies that $p(\alpha) = 0$. The set of *roots* is $R = Re \cup Im$. For $\lambda \in \mathbb{C}^I$ we set $R_\lambda = \{\alpha \in R : \lambda \cdot \alpha = 0\}$.

There is an immediate application of roots to counting simple representations of Π_λ.

Lemma 2.27. *[16, page 260] Let β be a dimension vector of a simple representation of Π_λ. If $p(\beta) = 0$ then there is a unique simple representation of Π_λ with dimension vector β (up to isomorphism); if $p(\beta) > 0$ then there are infinitely many non-isomorphic simple representations with dimension vector β.*

We return to representation type strata introduced earlier. For a representation type $\tau = (k_1, \beta^{(1)}; \ldots; k_r, \beta^{(r)})$, Crawley-Boevey proved the following

Proposition 2.28. *[16, Theorem 1.3] If $\mathcal{R}_\tau \neq \emptyset$ then $\mathcal{R}_\tau$ is an irreducible locally closed subset of $\mathcal{N}(\lambda, \alpha)$ of dimension $\sum_{t=1}^{r} 2p(\beta^{(t)})$.*

In order to understand $\mathcal{N}(\lambda, \alpha)$ further we shall need to describe the dimension vectors of simple Π_λ-modules.

Theorem 2.29. *[16, Corollary 1.4] For $\lambda \in \mathbb{C}^I$ and $\alpha \in \mathbb{N}^I$ the following are equivalent:*

1. *There is a simple representation of Π_λ of dimension vector α;*
2. *α is a positive root. $\lambda \cdot \alpha = 0$, and $p(\alpha) > \sum_{t=1}^{r} p(\beta^{(t)})$ for any decomposition $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$ with $r \geq 2$ and $\beta^{(t)}$ a positive root with $\lambda \cdot \beta^{(t)} = 0$ for all t.*

In this case $\mathcal{N}(\lambda, \alpha)$ is an irreducible variety of dimension $2p(\alpha)$ and its general element is a simple representation of Π_λ.

Henceforth we will let Σ_λ denote the set of dimension vectors of irreducible representations of Π_λ, that is, vectors satisfying condition (2) of the above Theorem.

The following theorem of Crawley-Boevey reduces the study of general $\mathcal{N}(\lambda, \alpha)$ to the simple case above. Let $\lambda \in \mathbb{C}^I$ and let $\mathbb{N}R_\lambda^+$ denote the set of sums (including zero) of the elements of the set R_λ^+ of positive roots β with $\beta \cdot \lambda = 0$. Let $\alpha \in \mathbb{N}R_\lambda^+$. Define

$$|\alpha|_\lambda = \max\left\{\sum_{t=1}^{r} p(\beta^{(t)}) : \alpha = \sum_{t=1}^{r} \beta^{(t)} \text{ with } \beta^{(t)} \in \Sigma_\lambda \text{ for all } t\right\}.$$

For any variety, X, there is an action of S_n on the n-fold product $X \times \cdots \times X$. The *nth symmetric product* of X is the variety $(X \times \cdots \times X)/S_n$ and we denote this by $\operatorname{Sym}^n X$.

Theorem 2.30. *[17, Theorem 1.1] Let $\lambda \in \mathbb{C}^I$ and $\alpha \in \mathbb{N}R_\lambda^+$.*

1. *There is a unique decomposition $\alpha = \sigma^{(1)} + \cdots \sigma^{(r)}$ with $\sigma^{(t)} \in \Sigma_\lambda$ for all t, such that $|\alpha|_\lambda = \sum_{t=1}^{r} p(\sigma^{(t)})$.*

2. *Any other decomposition of α as a sum of elements of Σ_λ is a refinement of this decomposition.*

3. *Collecting terms and rewriting this decomposition as $\alpha = \sum_{t=1}^{s} m_t\sigma^{(t)}$ where $\sigma^{(1)}, \dots, \sigma^{(s)}$ are distinct and $m_1, \dots, m_s$ are positive integers, we have*
$$\mathcal{N}(\lambda, \alpha) \cong \prod_{t=1}^{s} \operatorname{Sym}^{m_t} \mathcal{N}(\lambda, \sigma^{(t)}).$$
In particular $\mathcal{N}(\lambda, \alpha)$ is irreducible of dimension $2|\alpha|_\lambda$.

Extended Dynkin diagrams

We give a useful lemma which applies to the quivers which will be of interest to us. Let Q be an extended Dynkin diagram (see Section 1.4), oriented to have no cycles (this is no restriction since the deformed preprojective algebras are orientation independent).

Since Q is an extended Dynkin quiver there is a unique root, $\delta \in Im^+$ which is minimal for the partial ordering on $\mathbb{Z}^I$. Recall that in the discussion following Notation 1.22 we described how Q could be formed from the McKay graph of a finite subgroup G of $\mathrm{SL}(2, \mathbb{C})$. If $S_0, \dots, S_k$ are the irreducible representations of G then $\delta_i = \dim_{\mathbb{C}} S_i$. Then δ is isotropic, that is, $(\delta, \delta) = 0$ and furthermore $(\delta, \epsilon_k) = 0$ for all $k \in I$. Furthermore, any vector α such that $(\alpha, \epsilon_i) = 0$ for all i is multiple of δ. Any vertex for which $\delta_k = 1$ is called an *extending vertex* - extending vertices always exist. We can relabel the vertex set I so that 0 is an extending vertex. All real roots for Q satisfy $(\alpha, \alpha) = 1$.

Let Q' be the quiver obtained from Q by adding a vertex ∞ and one arrow from 0 to this vertex. We'll use apostrophes to denote data associated with Q'.

Lemma 2.31. *Let $n \in \mathbb{N}$. Then*

(1) $p'(\epsilon_\infty + n\delta) = n$.

(2) If $\epsilon_\infty + n\delta = \beta^{(1)} + \cdots + \beta^{(r)}$, where the $\beta^{(i)}$ are positive roots for Q', then $\sum_{t=1}^{r} p'(\beta^{(t)}) \leq p'(\epsilon_\infty + n\delta)$, with equality exactly when all but one of the $\beta^{(t)}$ are equal to δ.

Proof. (1) This is a direct calculation, $p'(\epsilon_\infty + n\delta) = 1 - \frac{1}{2}((\epsilon_\infty, \epsilon_\infty) + 2(n\delta, \epsilon_\infty) + (n\delta, n\delta)) = 1 - \frac{1}{2}(2 - 2n + 0) = n$.

(2) This is [16, Lemma 9.2].

□

2.4 The local normal form and hyper-Kähler manifolds

We define the local normal form for a moment map which is the construction which will allow us to describe the symplectic leaves for Marsden-Weinstein reductions for quivers. Although the existence of the local normal form can be established under fairly mild assumptions in the case of real symplectic manifolds, the extra rigidity imposed by a hyper-Kähler structure is required in the complex situation.

The local normal form for the moment map

Suppose that (M, ω) is a symplectic variety on which a connected reductive algebraic group G acts symplectically, and suppose that the action is Hamiltonian. Let μ be the corresponding moment map. We review a construction discussed in [63, Section 3.2]. We wish to consider a closed orbit corresponding to a point in $\mu^{-1}(0)//G$, say Gm, for some $m \in \mu^{-1}(0)$. Let H be the stabiliser of m, and denote its Lie algebra by $\mathfrak{h}$. Then H is a reductive group by Lemma 2.5. Let $T_m(Gm)$ be the tangent space of the point m in Gm, and let $(T_m(Gm))^{\omega} = \{v \in T_m M : \omega(v, w) = 0 \text{ for all } w \in T_m(Gm)\}$. By [36, page 324] $T_m(Gm) \subseteq (T_m(Gm))^{\omega}$. Set $\hat{M} = (T_m(Gm))^{\omega}/T_m(Gm)$. The vector bundle, $(G \times \hat{M})//H$, is called the *symplectic normal bundle*. The action of H preserves the induced symplectic structure on $\hat{M}$. Let $\hat{\mu} : \hat{M} \to \mathfrak{h}^*$ be the corresponding moment map from Theorem 2.18 which is H-equivariant.

The idea of the local normal form for the moment map is to provide a model space which is locally (in the complex topology) equivalent to a neighbourhood of $G/H \cong Gm$ in $\mu^{-1}(0)//G$. We choose an $\mathrm{Ad}(H)$-invariant splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ and a dual splitting $\mathfrak{g}^* = \mathfrak{h}^* \oplus \mathfrak{h}^{\perp *}$. Then we consider the H-action on $G \times \mathfrak{h}^{\perp *} \times \hat{M}$ by $a \cdot (g, \xi, \hat{m}) = (ga, \mathrm{Ad}^*(a^{-1})\xi, a^{-1}\hat{m})$ for $a \in H$, and where $\mathrm{Ad}^*(a)$ denotes the coadjoint representation of a on $\mathfrak{g}^*$. The quotient $(G \times \mathfrak{h}^{\perp *} \times \hat{M})//H$ is the associated bundle of the action of H on $\mathfrak{h}^{\perp *} \times \hat{M}$ so is smooth by Proposition 2.6.

It is locally trivial so the tangent space at each point can be identified with the tangent space of a point in $G/H \times \mathfrak{h}^{\perp *} \times \hat{M}$ which is equal to $(\mathfrak{h}^{\perp} \oplus \mathfrak{h}^{\perp *}) \oplus \hat{M}$. Therefore $(G \times \mathfrak{h}^{\perp *} \times \hat{M})//H$ is a symplectic manifold and there is a natural embedding:

$$G/H = G/H \times 0 \times 0 \hookrightarrow (G \times \mathfrak{h}^{\perp *} \times \hat{M})//H.$$

One can calculate the symplectic normal bundle of G/H in $(G \times \mathfrak{h}^{\perp *} \times \hat{M})//H$ using Lemma 2.7, and it equals $(G \times \hat{M})//H$.

There is an action of G on $(G \times \mathfrak{h}^{\perp *} \times \hat{M})//H$ by $a \cdot [g, \xi, \hat{m}] = [ag, \xi, \hat{m}]$ for all $a \in G$. This action is Hamiltonian with corresponding G-invariant moment map

$$J : (G \times \mathfrak{h}^{\perp *} \times \hat{M})//H \to \mathfrak{g}^*;\ [g, \xi, \hat{m}] \mapsto \mathrm{Ad}^*(g)(\xi + \hat{\mu}(\hat{m})).$$

Definition 2.32. *We say that the local normal form for μ exists if there is a G-invariant holomorphic bijection, intertwining symplectic structures, between a neighbourhood of Gm in M and a neighbourhood of G/H in $(G \times \mathfrak{h}^{\perp *} \times \hat{M})//H$.*

In examples we are interested in the local normal form will exist due to some additional structure on M, namely M will be a hyper-Kähler manifold.

Hyper-Kähler manifolds

We review briefly the notion of a hyper-Kähler manifold, see [39] or [62] for more details. Given a *real* manifold, M, a *complex structure* on M is a smooth map $I : TM \to TM$ which, for each $m \in M$, induces a linear map $I_m : T_mM \to T_mM$ such that $I_m^2 = -\mathrm{Id}$.

Definition 2.33. *A hyper-Kähler manifold is a $4n$-dimensional real manifold, M, with a Riemannian metric, g, and three complex structures, I, J and K, satisfying:*

1. *for each $m \in M$, I_m, J_m and $K_m \in \mathrm{O}(T_mM)$, the orthogonal group of T_mM;*
2. *for each $m \in M$, $I_m^2 = J_m^2 = K_m^2 = I_mJ_mK_m = -\mathrm{Id}$;*
3. *the nondegenerate real 2-forms on M given by $\omega_1(v,w) = g(v, Iw)$, $\omega_2(v,w) = g(v, Jw)$ and $\omega_3(v,w) = g(v, Kw)$ for all $v, w \in TM$ are closed.*

Condition (3) is usually stated in terms of I, J and K being parallel with respect to the Levi-Civita connection of g; the two formulations are equivalent by [62, Lemma 3.3.7]. If we consider M with the complex structure I then this makes M into a $2n$-dimensional complex manifold. Thus if we define $\omega = \omega_2 + \sqrt{-1}\omega_3$ then, with respect to I, this is a complex nondegenerate closed 2-form.

Example 2.34 (Representations of quivers). We describe the hyper-Kähler structure for representations of (doubles of) quivers, further details can be found in [62, Section 3] and [61, Section 2]. For a quiver Q and dimension vector α, we consider the (real) manifold $\mathrm{Rep}(\overline{Q},\alpha)$ with complex structure I given by multiplication by $\sqrt{-1}$ (since $\mathrm{Rep}(\overline{Q},\alpha)$ is a vector space we identify the tangent space at each point with the space itself). A Riemannian metric on $\mathrm{Rep}(\overline{Q},\alpha)$ is simply a real inner product, and we require that this is invariant under I. This is equivalent to having a Hermitian inner product on $\mathrm{Rep}(\overline{Q},\alpha)$, and one recovers the real inner product by taking the real part of the Hermitian form.

For each $i \in I$ we give each $\mathbb{C}^{\alpha_i}$ the standard Hermitian inner product. We get a Hermitian inner product on $\mathrm{Mat}(\alpha_j \times \alpha_i)$ by $(B,C) = tr(BC^\dagger)$ for all $B, C \in \mathrm{Mat}(\alpha_j \times \alpha_i)$ (we use $\dagger$ to denote the Hermitian conjugate). This induces a Hermitian form on $\mathrm{Rep}(\overline{Q},\alpha)$. We get further complex structures by

$$J(\underline{B}, \underline{B}^*) = ((B_{a^*}^\dagger), (-B_a^\dagger)),\ K = -JI$$

and this makes $\mathrm{Rep}(\overline{Q},\alpha)$ into a hyper-Kähler manifold (since the 2-forms ω_i are symplectic forms on a vector space they are closed). Let $a \in A$ and consider $M \in \mathrm{Mat}(\alpha_{h(a)} \times \alpha_{t(a)}, \mathbb{C}), N \in \mathrm{Mat}(\alpha_{h(a^*)} \times \alpha_{t(a^*)}, \mathbb{C})$ as elements of $\mathrm{Rep}(\overline{Q},\alpha)$. Then

$$(\omega_2 + \sqrt{-1}\omega_3)(M,N) = \mathrm{Re}\ \mathrm{tr}(M(JN)^\dagger) + \sqrt{-1}\mathrm{Re}\ \mathrm{tr}(M(-JIN)^\dagger) =$$
$$\mathrm{Re}\ \mathrm{tr}(M(N^\dagger)^\dagger) - \sqrt{-1}\mathrm{Re}\ \mathrm{tr}(M((\sqrt{-1}N)^\dagger)^\dagger) = \mathrm{Re}\ \mathrm{tr}(MN) - \sqrt{-1}\mathrm{Re}\ \sqrt{-1}\,\mathrm{tr}(MN) =$$
$$\mathrm{Re}\ \mathrm{tr}(MN) + \sqrt{-1}\mathrm{Im}\ \mathrm{tr}(MN) = \mathrm{tr}(MN).$$

Therefore the definition of ω above and that in Section 2.3 agree.

Let $U(n)$ be the subgroup of $\mathrm{GL}_n(\mathbb{C})$ of unitary matrices. Let $K(\alpha) = \prod_{k\in I} U(\alpha_i)/U(1)$, a compact connected (real) Lie group acting on $\mathrm{Rep}(\overline{Q},\alpha)$ preserving the Hermitian form. The action of this group preserves the real symplectic form ω_1 and we can work out the corresponding moment map (defined analogously to moment maps for complex symplectic varieties, Section 2.2), $\mu_1 : \mathrm{Rep}(\overline{Q},\alpha) \to \mathfrak{k}(\alpha)^*$ where $\mathfrak{k}(\alpha) = \mathrm{Lie}\ K(\alpha)$:

$$\mu_1(\underline{B}_a, \underline{B}_{a^*}) = \frac{\sqrt{-1}}{2}\Big(\sum_{a\in A; h(a)=k} B_a B_a^\dagger - B_{a^*}^\dagger B_{a^*}\Big)_k.$$

On the other hand we can think of $\mathrm{G}(\alpha)$ as the complexification of $K(\alpha)$, and this acts on $\mathrm{Rep}(\overline{Q},\alpha)$ preserving ω and the corresponding (complex) moment map is the map μ_α defined in Section 2.3.

Proposition 2.35. *[43, Theorem 0.2] Let V be a complex vector space with Hermitian inner product, $(-,-)$. Let K be a closed connected (real) subgroup of $U(V)$ and let G be the complexification of K. Let ω_1 be the symplectic form given by $\omega_1(v,w) = (v, \sqrt{-1}w)$ and μ_1 the canonical moment map corresponding to ω_1. Then a G-orbit in V is closed if and only if it intersects $\mu_1^{-1}(0)$ nontrivially.*

In particular the above proposition applies to $V = \mathrm{Rep}(\overline{Q}, \alpha)$, $K = \prod_{k \in I} U(\alpha_i)$ and $G = \prod_{k \in I} \mathfrak{gl}(\alpha_i)$ and where ω_1, μ_1 are as above (of course, we really want to consider $K(\alpha)$ and $G(\alpha)$ acting on V, but since these are obtained from the K and G above by factoring out subgroups which act on V trivially, the result extends easily to our case). Recall the definition of the local normal form given in 2.4.

Corollary 2.36. *The local normal form for μ_α exists.*

Proof. This is [63, Lemma 3.2.1]. $\square$

We shall give some details in order to make clear the ingredients of the proof of Corollary 2.36. Let $\Upsilon = (G \times \mathfrak{h}^{\perp *} \times \hat{M})//H$ and $M = \mathrm{Rep}(\overline{Q}, \alpha)$. The embeddings of $G/H \cong Gm$ in Υ and M respectively have the same symplectic normal bundles. The G-equivariant version of the Darboux-Moser-Weinstein isotropic embedding theorem, [74, Theorem 2.2], says that the local normal form exists in this situation when our manifold, M, is a *real* manifold and G is a compact group. The proof is based on the inverse function theorem, which is true also in the category of complex manifolds. The obstruction to extending the result to our situation is that the domain of the holomorphic bijection may not cover the whole of Gm, because it is not compact. However, the orbit Km is compact so we can give the K-equivariant version of the local normal form, namely that there exists a K-stable open neighbourhood of $m \in M$ which maps isomorphically, symplectically and K-equivariantly to a neighbourhood of the image of m in Υ.

The method for extending this is based on a result in [73]. Let K be a compact real Lie group, and let G be its complexification. A subset A of a G-space X is said to be *orbitally convex* with respect to the G-action if it is invariant under K and for all $x \in A$, and all $\xi \in \mathfrak{k}$ we have that both x and $\exp(\sqrt{-1}\xi)x$ are in A implies that $\exp(\sqrt{-1}t\xi)x \in A$ for all $t \in [0,1]$.

The reason we want to consider such sets is their extension properties.

Proposition 2.37. *[73, Proposition 1.4] Let X and Y be complex manifolds acted on by G (in the notation above). If A is an orbitally convex open subset of X and $f : A \to Y$ is a K-equivariant holomorphic map, then f can be uniquely extended to a G-equivariant holomorphic map $\tilde{f} : GA \to Y$.*

If f is an isomorphism onto its image, $f(A)$, then $\tilde{f} : GA \to Gf(A)$ is an isomorphism.

Thus, in order to prove Corollary 2.36, we need to establish that any K-stable open set of M containing Km is orbitally convex. Recall that M is a real symplectic manifold with form ω_1. We say that a real submanifold, N, of M is *isotropic* if $\omega_1(n)(T_nN, T_nN) = 0$ for all $n \in N$. It is a standard result from symplectic geometry that if $\xi \in \mathfrak{k}^*$ is a fixed point then for any $x \in \mu_1^{-1}(\xi)$, Kx is isotropic, [36, page 324]. In the case of M we have chosen some m such that Gm is a closed orbit. Therefore by Proposition 2.35, we can assume that $\mu_1(m) = 0$ and that Km is isotropic.

The next result is generally true for isotropic orbits in a Kähler manifold with the action of a reductive complex Lie group which is the complexification of a compact group. M is a Kähler manifold via the form ω_1 and the complex structure I, see [55] for a definition of Kähler manifolds.

Proposition 2.38. *[73, Claim 1.13] The orbit Km has a basis of orbitally convex open heighbourhoods, that is, any K-stable neighbourhood of Km is orbitally convex.*

2.5 Remarks

1. The invariant theory we mention is standard material, see [68], [48] or [75] for a deeper introduction.

2. Presumably Example 2.8 is well-known.

3. Proposition 2.12 is not stated explicitly in [3] but it follows easily from the discussion therein.

4. Much of our material on moment maps and reduction comes from [15]; a complementary article is [46].

5. Facts about quivers and roots were obtained from [16], and the particular case of extended Dynkin diagrams is covered in [19].

Chapter 3

Associated varieties

We prove that the associated variety of a Poisson prime ideal of $Z_{0,c}$ is irreducible, Theorem 3.7. This result has direct analogues in earlier theorems for complex semisimple Lie algebras and for the $t = 1$ case of symplectic reflection algebras, see Corollary 3.8 and Corollary 3.9 respectively. Our proof of Theorem 3.7 is based closely on the proof of [29, Theorem 2.1] which in turn is based on the arguments used in [78, § 3-4]. We discuss the relevance of our theorem for studying symplectic leaves at the end of Section 3.2. Section 3.3 contains definitions and known results. Most of this chapter has been published in [57].

3.1 Preliminaries

Throughout this chapter, when we consider the variety associated to a finitely generated commutative algebra, R, we shall think of its reduced scheme structure, that is, $\operatorname{Spec} R$ rather than $\operatorname{Max} R$. Let A be a $\mathbb{C}$-algebra with identity element e. We shall say that a $\mathbb{Z}$-filtration, $\mathcal{F}$, of A is *suitable* when we have:

$$0 = \mathcal{F}_{-1} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq A$$

and $\mathcal{F}$ satisfies

(*a*) $\mathcal{F}_i \cdot \mathcal{F}_j \subseteq \mathcal{F}_{i+j}$, (*b*) $\mathcal{F}_0 = \mathbb{C}e$, (*c*) $\mathrm{Dim}_{\mathbb{C}}\mathcal{F}_\mathrm{i} < \infty$ for all i and (d) $\mathrm{gr}^{\mathcal{F}}$ A is an affine commutative $\mathbb{C}-$ algebra.

Example 3.1. Taking $eH_{t,c}e$ with filtration $\mathcal{E}$, defined in Section 1.3, gives an example of a suitable filtration.

Definition 3.2. *Let A be a $\mathbb{C}$-algebra with suitable filtration, $\mathcal{F}$. We say that A has a proto-Poisson bracket with respect to $\mathcal{F}$ if there exists a non-zero skew-symmetric $\mathbb{C}$-bilinear map $\langle -,- \rangle : A \times A \to A$ which satisfies, for all $a, b, c \in A$:*

1. *$\langle ab, c\rangle = a\langle b, c\rangle + \langle a, c\rangle b$.*

2. *$\langle a, \langle b, c\rangle\rangle = \langle\langle a, b\rangle, c\rangle + \langle b, \langle a, c\rangle\rangle$.*

3. *There is an integer d, the degree of $\langle -,- \rangle$, such that $\langle \mathcal{F}_i, \mathcal{F}_j\rangle \subseteq \mathcal{F}_{i+j+d}$ for all $i, j \in \mathbb{Z}$, but there exist $i, j \in \mathbb{Z}$ such that $\langle \mathcal{F}_i, \mathcal{F}_j\rangle \nsubseteq \mathcal{F}_{i+j+d-1}$.*

Example 3.3.

(i) Let A be an algebra with suitable filtration $\mathcal{F}$. Suppose that A is generated by $\mathcal{F}_1$ with $\mathcal{F}_i = (\mathcal{F}_1)^i$, and that A is not commutative. Let $\langle -,- \rangle$ equal the commutator bracket on A. Then $\langle -,- \rangle$ is a proto-Poisson bracket on A. The only nontrivial condition to check is (3). For this let d be the integer such that $\langle \mathcal{F}_1, \mathcal{F}_1\rangle \subseteq \mathcal{F}_{d+2}$ but $\langle \mathcal{F}_1, \mathcal{F}_1\rangle \nsubseteq \mathcal{F}_{d+1}$. There exists such an integer because A is not commutative. It is easily seen that $\langle \mathcal{F}_i, \mathcal{F}_j\rangle \subseteq \mathcal{F}_{i+j+d}$ for all $i, j \in \mathbb{Z}$.

(ii) Let R be a Poisson algebra with suitable filtration $\mathcal{F}$. Let $\{-,-\}$ be the Poisson bracket on R. If $\{-,-\}$ satisfies condition (3) of the definition then it is a proto-Poisson bracket. In particular, for any finitely generated Poisson algebra, R, with generating set $\{a_1, \dots, a_t\}$ we can define a filtration, $\mathcal{F}$, by $\mathcal{F}_i = 0$ for $i < 0$, $\mathcal{F}_0 = \mathbb{C}e$, $\mathcal{F}_1 = \mathbb{C}e + \sum_{i=1}^{t} \mathbb{C}a_i$ and $\mathcal{F}_i = (\mathcal{F}_1)^i$ for $i \geq 2$. Then $\mathcal{F}$ is a suitable filtration and the Poisson bracket on R is a proto-Poisson bracket. In particular, it follows from the Leibniz rule that condition (3) will hold for d, where $d = \min\{k : \{a_i, a_j\} \in \mathcal{F}_k \text{ and } \{a_i, a_j\} \notin \mathcal{F}_{d-1}, 1 \leq i, j \leq t\}$.

Lemma 3.4. *Consider $eH_{t,c}e$ with suitable filtration $\mathcal{E}$.*

(i) When $t = 1$ the commutator is a proto-Poisson bracket of degree -2.

(ii) When $t = 0$ the Poisson bracket is a proto-Poisson bracket of degree -2.

Proof.

(i) The only condition of Definition 3.2 which is non-trivial is (3), but this follows from [26, Claim 2.25(i)], where it is also proved that the degree, d, is -2 in this case.

(ii) Let $\{-,-\}$ be the Poisson bracket on $eH_{0,c}e$. As noted in Examples 3.3 (ii), we need to show that condition (3) of Definition 3.2 is satisfied. It can be seen from the construction of $\{-,-\}$ that there is some $l \geq 2$ so that $\{\mathcal{F}_i, \mathcal{F}_j\} \subseteq \mathcal{F}_{i+j-l}$, for all i, j $\in \mathbb{Z}$ but $\{\mathcal{F}_i, \mathcal{F}_j\} \nsubseteq \mathcal{F}_{i+j-(l+1)}$, for some i, j $\in \mathbb{Z}$. By [26, Lemma 2.26], we must have $l = 2$.

$\square$

We shall say that an ideal I of A is a $\langle -,- \rangle$-*ideal* if $\langle A, I \rangle \subseteq I$. In Example 3.3 (i), a $\langle -,- \rangle$-ideal is simply an ideal; in Example 3.3 (ii), a $\langle -,- \rangle$-ideal is just a Poisson ideal.

Lemma 3.5. *Let A be a $\mathbb{C}$-algebra with suitable filtration, $\mathcal{F}$, and proto-Poisson bracket, $\langle -,- \rangle$ of degree d. For homogeneous elements $x, y \in \operatorname{gr} A$ of degree k and l respectively, denote lifts of x and y by $\tilde{x}, \tilde{y} \in A$, that is, $\sigma_k(\tilde{x}) = x$ and $\sigma_l(\tilde{y}) = y$ where σ denotes the principal symbol map. Then*

$$\operatorname{gr}\langle -,- \rangle : \operatorname{gr} A \times \operatorname{gr} A \to \operatorname{gr} A; \quad (x, y) \mapsto \sigma_{i+j+d}(\langle \tilde{x}, \tilde{y} \rangle)$$

defines a Poisson bracket of degree d on $\operatorname{gr} A$ when extended linearly.

Proof. That $\operatorname{gr}\langle -,- \rangle$ is a Poisson bracket follows directly from the definition of a proto-Poisson bracket. It is straightforward to see that the degree of $\langle -,- \rangle$ equals the degree of $\operatorname{gr}\langle -,- \rangle$. $\square$

3.2 Statement and applications

We state the main theorem of this chapter and give some applications. Corollaries 3.8 and 3.9 are known results which follow easily from Theorem 3.7, but our main interest is in Corollary 3.11.

We begin with a straightforward lemma.

Lemma 3.6. *Let R be an affine commutative $\mathbb{C}$-algebra with two Poisson brackets, $\{-,-\}_1$ and $\{-,-\}_2$, such that $\{-,-\}_1 = \lambda\{-,-\}_2$ for some non-zero $\lambda \in \mathbb{C}$. Let $X =$ Spec R. Then the symplectic leaves of X with respect to $\{-,-\}_1$ are the same as the symplectic leaves of X with respect to $\{-,-\}_2$. In particular, $(X, \{-,-\}_1)$ has finitely many symplectic leaves if and only if $(X, \{-,-\}_2)$ has finitely many symplectic leaves.*

Proof. The rank of $\{-,-\}_1$ at any closed point $\mathfrak{m}$ of X is equal to the rank of $\{-,-\}_2$ at $\mathfrak{m}$, so the lemma follows from the definition of symplectic leaf. $\square$

Theorem 3.7. *Let A be a $\mathbb{C}$-algebra with suitable filtration, $\mathcal{F}$, and proto-Poisson bracket $\langle -,- \rangle$. Let* $\mathrm{gr}^{\mathcal{F}} A$ *have Poisson bracket* $\mathrm{gr}\langle -,- \rangle$ *and let I be a prime $\langle -,- \rangle$-ideal of A. Suppose* $X =$ Spec $\mathrm{gr}^{\mathcal{F}}\mathrm{A}$ *has finitely many symplectic leaves with respect to the Poisson bracket induced on it by $\langle -,- \rangle$. Let $Y = \mathcal{V}(\mathrm{gr}^{\mathcal{F}} I)$. Then Y is irreducible, and is the closure of a symplectic leaf in X.*

The second claim follows quickly from the first. For suppose that Y is irreducible. Since I is a $\langle -,- \rangle$-ideal it can easily be seen that $\mathrm{gr}^{\mathcal{F}} I$ is a Poisson ideal, and therefore $\mathrm{rad}(\mathrm{gr}^{\mathcal{F}} I)$ is also Poisson by [23, 3.3.2]. Hence Y is a closed irreducible Poisson subvariety and is the closure of a symplectic leaf in X, by Theorem 1.43 (ii). We note that a version of Theorem 3.7 is true with the weaker assumption that Y (and not X) has finitely many symplectic leaves. Then Y is irreducible, but is not necessarily the closure of a symplectic leaf of X.

We get as a corollary to Theorem 3.7 a proof of the result of Borho and Brylinski, [7], and Joseph, [42]. There is a detailed account closer in spirit to the original proofs, including the background material required, in [78, §3 – 4].

Corollary 3.8. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\mathcal{U}(\mathfrak{g})$ denote its enveloping algebra. Then the associated variety of a primitive ideal of $\mathcal{U}(\mathfrak{g})$ is irreducible.*

Proof. Let $[-,-]$ denote the Lie bracket of $\mathfrak{g}$. There is suitable filtration, $\mathcal{B}$, on $\mathcal{U}(\mathfrak{g})$ where $\mathcal{B}_1 = \mathfrak{g}$ generates $\mathcal{U}(\mathfrak{g})$ as an algebra and $\mathcal{B}_i = (\mathcal{B}_1)^i$ for all $i \geq 1$. Now $\mathrm{gr}\mathcal{U}(\mathfrak{g}) = S\mathfrak{g}$, and the variety Spec $\mathrm{gr}\mathcal{U}(\mathfrak{g})$ can be identified with $\mathfrak{g}^*$. As explained in Example 3.3 (i), setting $\langle -,- \rangle$ equal to the commutator on $\mathcal{U}(\mathfrak{g})$ defines a proto-Poisson bracket on $\mathcal{U}(\mathfrak{g})$. Therefore there is a Poisson bracket, $\mathrm{gr}\langle -,- \rangle$, on $\mathrm{gr}\mathcal{U}(\mathfrak{g})$. However, since $\mathrm{gr}\mathcal{U}(\mathfrak{g}) = S\mathfrak{g}$, it is clear that $\mathrm{gr}\langle -,- \rangle$ is extended from the Lie bracket on $\mathfrak{g}$, giving the Kostant-Kirillov Poisson bracket of Example 1.28. Let P be a primitive ideal of $\mathcal{U}(\mathfrak{g})$, and let Q be a minimal primitive ideal contained in P. Now, $\mathfrak{g}^*$ will never have finitely many symplectic leaves, but the Poisson subvariety $\mathcal{V}(\mathrm{gr}Q)$ does (by [78, Theorem 5.8]), the leaves being the nilpotent coadjoint orbits (see [55, Theorem 14.3.1]). If we now take $A = \mathcal{U}(\mathfrak{g})/Q$ with filtration and proto-Poisson bracket induced from $\mathcal{U}(\mathfrak{g})$, Theorem 3.7 tells us that $\mathcal{V}(\mathrm{gr}P)$ is an irreducible subvariety of $\mathcal{V}(\mathrm{gr}Q)$ and therefore also of $\mathfrak{g}^*$. ☐

We get as another corollary the following result which was proved by Ginzburg in [29, Theorem 2.1].

Corollary 3.9. *For any primitive ideal I of $eH_{1,c}e$, the variety $\mathcal{V}(\mathrm{gr}^{\mathcal{E}}\mathrm{I})$ is irreducible.*

Proof. Let $[-,-]$ denote the commutator bracket on $eH_{1,c}e$. By Lemma 3.4 (i), $\mathrm{gr}[-,-]$ is a Poisson bracket of degree -2 on $\mathrm{gr}^{\mathcal{E}}\, eH_{1,c}e \cong SV^G$ and so by [26, Lemma 2.23(i)] there is some non-zero $\lambda \in \mathbb{C}$ such that $\mathrm{gr}[-,-] = \lambda\{-,-\}_\omega$. By Lemma 3.6 and Theorem 1.44, Spec $\mathrm{SV^G}$, with Poisson bracket $\mathrm{gr}[-,-]$, has finitely many symplectic leaves. We can now apply Theorem 3.7: for any prime ideal I of $eH_{1,c}e$, $\mathcal{V}(\mathrm{gr}^{\mathcal{E}}\mathrm{I})$ is irreducible. In particular, this is true for any primitive ideal I. □

We extend this to the case when $t = 0$.

Corollary 3.10. *For any Poisson prime ideal, I, of $eH_{0,c}e$, $\mathcal{V}(\mathrm{gr}^{\mathcal{E}}\mathrm{I})$ is irreducible.*

Proof. Let $\{-,-\}$ denote the Poisson bracket on $eH_{0,c}e$. By Lemma 3.4 (ii), $\{-,-\}$ is a proto-Poisson bracket of degree -2. By [26, Lemma 2.23], $\mathrm{gr}\{-,-\} = \lambda\{-,-\}_\omega$ for some $\lambda \in \mathbb{C}^*$. Therefore by Theorem 1.44 and Lemma 3.6, V/G, with Poisson bracket $\mathrm{gr}\{-,-\}$ has finitely many symplectic leaves. Hence $\mathcal{V}(\mathrm{gr}^{\mathcal{E}}\mathrm{I})$ is irreducible. □

Finally we consider $Z_{0,c}$.

Corollary 3.11. *Let I be a Poisson prime ideal of $Z_{0,c}$. Then $\mathcal{V}(\mathrm{gr}^{\mathcal{Z}}\mathrm{I})$ is irreducible.*

Proof. Let ψ denote the Satake isomorphism. Let P be a prime ideal of $Z_{0,c}$, then by Theorem 1.31, P is Poisson if and only if $\psi(P)$ is Poisson. Furthermore, by Proposition 1.14, $\mathcal{V}(\mathrm{gr}^{\mathcal{Z}}\mathrm{P})$ is irreducible if and only if $\mathcal{V}(\mathrm{gr}^{\mathcal{E}}\psi(\mathrm{P}))$ is irreducible, and, by Corollary 3.10, $\mathcal{V}(\mathrm{gr}^{\mathcal{E}}\psi(\mathrm{P}))$ is irreducible.

□

We spell out the implications of this last result for the symplectic leaves of $X_c = \operatorname{Spec} Z_c$. When we take any symplectic leaf $\mathcal{S}$ in X_c then $\bar{\mathcal{S}} = \mathcal{V}(P)$ for some Poisson prime ideal of Z_c (Theorem 1.43 (i)). Now Corollary 3.11 tells us that the associated variety of P is the closure of a symplectic leaf in V/G. However, the symplectic leaves of V/G are known and are described by conjugacy classes of subgroups of G which are the stabilisers of some element of V, Example 1.37. We therefore obtain a map

$$\Omega : \{\text{Symplectic leaves in } X_c\} \to \{\text{Conjugacy classes of subgroups in } G\}. \tag{3.1}$$

The properties of Ω should give further information about the symplectic leaves of X_c and perhaps also about the representation theory of H_c, cf. Theorem 1.44. We shall discuss this further in the remarks at the end of this chapter, see also Corollary 5.8.

3.3 Microlocalisation

We describe the process of microlocalisation which is detailed in [78, § 3-4] and will be used in the proof of Theorem 3.7. Throughout this section we assume that A is a $\mathbb{C}$-algebra with a $\mathbb{Z}$-filtration, $\mathcal{F}$. We shall say that a $\mathbb{Z}$-filtration, $\mathcal{B}$, of an A-module M is *compatible* if $\mathcal{F}_i\mathcal{B}_j \subseteq \mathcal{B}_{i+j}$ for all i, j. The filtration $\mathcal{B}$ is *good* if it is compatible, $\bigcap_{n\in\mathbb{Z}} \mathcal{B}_n = 0$, $\bigcup_{n\in\mathbb{Z}} \mathcal{B}_n = M$ and $\operatorname{gr}^{\mathcal{B}} M$ is a finitely generated $\operatorname{gr}^{\mathcal{F}} A$-module.

Suppose that $\operatorname{gr}^{\mathcal{F}} A$ is a commutative algebra and that $\mathcal{F}$ is a good filtration of A. Let I be an ideal of A and U be an open subset of $X = \operatorname{Spec} \operatorname{gr}^{\mathcal{F}} A$ defined by homogeneous elements, $f_1, \dots, f_t$, of $\operatorname{gr}^{\mathcal{F}} A$. In other words, $U = X \setminus \mathcal{V}(f_1, \dots, f_t)$ where $\mathcal{V}(f_1, \dots, f_t)$ denotes the closed subvariety corresponding to the ideal generated by the f_i.

We recall the definition of support. Let R be a commutative ring and let N be an R-module. Then

$$\operatorname{supp}_R N = \{P \in \operatorname{Spec} R : N_P \neq 0\}.$$

We write supp N when the ring is clear from the context. It is straightforward to show that when N is finitely generated then supp N = $\mathcal{V}$(ann N), [25, Corollary 2.7].

For $M = A/I$, consider the induced filtration on M (which we also call $\mathcal{F}$) - this is a good filtration of M. Then

$$\operatorname{supp} \operatorname{gr}^{\mathcal{F}} M = \mathcal{V}(\operatorname{ann} \operatorname{gr}^{\mathcal{F}} M) = \mathcal{V}(\operatorname{gr}^{\mathcal{F}} I) \tag{3.2}$$

where the second equality is true since $\operatorname{gr}^{\mathcal{F}} M \cong \operatorname{gr}^{\mathcal{F}} A / \operatorname{gr}^{\mathcal{F}} I$. Now $\operatorname{gr}^{\mathcal{F}} M$ defines a sheaf of $\mathcal{O}_X$-modules, $\mathcal{M}$, on X and we can calculate the sections of $\mathcal{M}$ over U explicitly:

$$\mathcal{M}(U) = \{(m_{f_i}) \in \prod_{i=1}^{t} (\operatorname{gr}^{\mathcal{F}} M)_{f_i} : m_{f_i} = m_{f_j} \in (\operatorname{gr}^{\mathcal{F}} M)_{f_i f_j} \forall i, j\}. \tag{3.3}$$

Lemma 3.12. *[78, Lemma 3.3] Let K be the kernel of the natural* $\operatorname{gr}^{\mathcal{F}} A$*-module map*

$$\beta : \operatorname{gr}^{\mathcal{F}} M \to \mathcal{M}(U);\ m \mapsto (m).$$

Then $K = \{m \in \operatorname{gr} M :$ for each i there exists N_i in $\mathbb{N}$ *such that ${f_i}^{N_i} m = 0\}$, and*
$(\operatorname{supp} K) \cap U = \emptyset$.

Microlocalisation introduces a new filtration, Γ, on M which is compatible with $\mathcal{F}$ [78, Corollary 6.9]. We can give a description of Γ in terms of the $f_i s$ introduced above. Let p_i be the degree of

f_i and for each i choose a lift, ϕ_i, of f_i to A. Thus $\phi_i \in \mathcal{F}_{p_i}$ and $\sigma_{p_i}(\phi_i) = f_i$, where σ_{p_i} denotes the principal symbol map.

Let $\mathcal{I} = \{1, \ldots, t\}$ and suppose $\tau = (i_1, \ldots, i_N) \in \mathcal{I}^N$ is an ordered N-tuple of elements of $\mathcal{I}$. Define

$$p_\tau = \sum_{j=1}^{N} p_{i_j}, \quad \phi_\tau = \prod_{j=1}^{N} \phi_{i_j} \in \mathcal{F}_{p_\tau}.$$

Then $\Gamma_n = \{m \in M : \text{ for all } N \text{ sufficiently large and for all } \tau \in \mathcal{I}^N, \phi_\tau \cdot m \in \mathcal{F}_{n+p_\tau}\}$. We see Γ has the property that $\mathcal{F}_i \subseteq \Gamma_i$ for all $i \in \mathbb{Z}$ so that $\mathrm{gr}^\Gamma M$ is a $\mathrm{gr}^\mathcal{F} A$-module and there is a canonical map (of $\mathrm{gr}^\mathcal{F} A$-modules) $\alpha : \mathrm{gr}^\mathcal{F} M \to \mathrm{gr}^\Gamma M$.

Proposition 3.13. *[78, Proposition 3.11] There exists a map of $\mathrm{gr}^\mathcal{F} A$-modules $\theta : \mathrm{gr}^\Gamma M \to \mathcal{M}(U)$ which is injective and gives rise to the following commutative diagram:*

$$\begin{tikzcd} \mathrm{gr}^\mathcal{F} M \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \mathrm{gr}^\Gamma M \arrow[d, "\theta"] \\ & \mathcal{M}(U) \end{tikzcd}$$

The following result will be key to the proof of Theorem 3.7: we will use it to prove that, in certain situations, $\mathcal{M}(U)$ is a finitely generated $\mathrm{gr}^\mathcal{F} A$-module.

Theorem 3.14. *[35, Proposition 5.11.1] Suppose R is an affine commutative algebra over $\mathbb{C}$, N is a finitely generated R-module and W is an open set in $\mathrm{Spec}R$. Let $\mathcal{N}$ denote the sheaf of modules associated to N. Then the R-module $\mathcal{N}(W)$ is finitely generated if and only if for every prime $P \in W \cap \mathrm{Ass}N$, $\overline{P}$, the closure of P in $\mathrm{Spec}R$, satisfies*

$$\overline{P} \cap (\mathrm{Spec}R \backslash W) \text{ has codimension at least 2 in } \overline{P}. \tag{3.4}$$

3.4 Proof of Theorem 3.7

Throughout this section we retain the hypotheses of Theorem 3.7. That is, let A be a $\mathbb{C}$-algebra with suitable filtration, $\mathcal{F}$, and proto-Poisson bracket $\langle -, - \rangle$. Let $\mathrm{gr}^\mathcal{F} A$ have Poisson bracket $\mathrm{gr}\langle -, - \rangle$ and let I be a prime $\langle -, - \rangle$-ideal of A. Suppose $X = \mathrm{Spec}\ \mathrm{gr}^\mathcal{F} \mathrm{A}$ has finitely many symplectic leaves with respect to $\mathrm{gr}\langle -, - \rangle$. Let $Y = \mathcal{V}(\mathrm{gr}^\mathcal{F} I)$ and let $M = A/I$. A closed subvariety of Y is said to be *homogeneous* if its defining ideal is a graded ideal of $\mathrm{gr}^\mathcal{F} A/\mathrm{gr}^\mathcal{F} I$; an open subvariety of Y is homogeneous if it is the complement of a closed homogeneous variety.

By [23, 3.3.2] the minimal primes of $\mathrm{gr}^{\mathcal{F}} A$ over $\mathrm{gr}^{\mathcal{F}} I$ are all Poisson ideals. Thus we can choose an irreducible component of Y of maximal dimension, and by Theorem 1.43 (ii) there exists a symplectic leaf $\mathcal{S}$ such that $\bar{\mathcal{S}}$ is this component and $\mathcal{S} = \mathrm{sm}\bar{\mathcal{S}}$. Then $\mathrm{Dim}\,\mathcal{S} = \mathrm{Dim}\,\bar{\mathcal{S}} = \mathrm{Dim}\,Y$, and by definition, the closed points of $\mathcal{S}$ all have rank equal to $\mathrm{Dim}\,Y$.

Lemma 3.15.

(1) $\mathcal{S}$ is open in Y.

(2) dim $\bar{\mathcal{S}} \setminus \mathcal{S} \leq \mathrm{dim}\bar{\mathcal{S}} - 2 = \mathrm{dimY} - 2$.

(3) $\mathcal{S}$ is a homogeneous subvariety of Y i.e. there exist homogeneous elements $g_1, \dots, g_s \in \mathrm{gr}^{\mathcal{F}} A / \mathrm{gr}^{\mathcal{F}} I$ such that $\mathcal{S} = Y \setminus \mathcal{V}(g_1, \dots, g_s)$.

Proof.

(1) Let $Y = I_1 \cup \cdots \cup I_k$ be an irredundant irreducible decomposition of Y with $I_1 = \bar{\mathcal{S}}$. We claim that $\mathcal{S} \cap I_j = \emptyset$ for all $2 \leq j \leq k$. If not, then for some j, $\mathcal{S} \cap I_j$ contains a closed point of rank $\mathrm{Dim}\,Y$, $\mathfrak{m}$ say. By Theorem 1.43 (ii) the smooth locus of I_j, $\mathrm{sm}I_j$, is a symplectic leaf in Y which contains $\mathfrak{m}$. Then $\mathcal{S} = \mathrm{sm}I_j$ implies that $I_1 = I_j$, a contradiction. Therefore $Y \setminus \mathcal{S} = (\bar{\mathcal{S}} \setminus \mathcal{S}) \cup I_2 \cup \cdots \cup I_k$ is closed in Y.

(2) It is clear from the definition of symplectic leaves that they are even dimensional. We can write $\bar{\mathcal{S}} \setminus \mathcal{S}$ as a finite union of symplectic leaves, each of which has dimension less than $\mathcal{S}$. The inequality follows because of even dimensionality.

(3) This is true because $\mathcal{S} = \mathrm{sm}I_1$. Since Y is homogeneous I_1 is also homogeneous so we may assume that the ideal of I_1 is generated by homogeneous elements $h_1, \dots, h_l$ in some polynomial ring $\mathbb{C}[x_1, \dots x_m]$. Now $\mathrm{sing}I_1$ is defined as the points vanishing at certain $(m - r) \times (m - r)$ minors of

$$\left(\partial h_i / \partial x_j \right).$$

These minors are homogeneous polynomials in the $x_j s$. Hence $\mathrm{sing}I_1$ and also $\mathrm{sm}I_1$ are homogeneous subvarieties.

□

Parts (1) and (3) of Lemma 3.15 imply that there is a homogeneous open set $U \subseteq X$ such that $U \cap Y = \mathcal{S}$. We write $U = X \setminus \mathcal{V}(f_1, \ldots, f_t)$ where the f_i are homogeneous elements of $\mathrm{gr}^{\mathcal{F}} A$.

We now turn to the microlocalisation techniques from Section 3.3. The $\mathrm{gr}^{\mathcal{F}} A$-module, $\mathrm{gr}^{\mathcal{F}} M$, defines a sheaf of modules, $\mathcal{M}$ on X. We only need to work with the sections, $\mathcal{M}(U)$ over the open set U from the previous paragraph. $\mathcal{M}(U)$ was described explicitly in (3.3). We also use the notation α, β and θ from Lemma 3.12 and Proposition 3.13. Recall that via microlocalisation one introduces a new filtration, Γ, which is compatible with $\mathcal{F}$.

We make the following two claims:

Claim 1. *Γ is a good filtration of M.*

Claim 2. $\mathrm{supp}_{\mathrm{gr}^{\mathcal{F}} A}\ \mathcal{M}(U) \subseteq \bar{\mathcal{S}}$.

Under the assumption that Claims (1) and (2) are true we prove Theorem 3.7.

Proof of Theorem 3.7. Now $\mathrm{supp}_{\mathrm{gr}^{\mathcal{F}} A}\ \mathrm{gr}^{\Gamma} M \subseteq \mathrm{supp}_{\mathrm{gr}^{\mathcal{F}} A}\ \mathcal{M}(U)$ because θ is injective. The left hand side equals Y by Claim (1) (see (3.2)) and the right hand side is contained in $\bar{\mathcal{S}}$ by Claim (2). So we have $Y \subseteq \bar{\mathcal{S}}$ which implies that $Y = \bar{\mathcal{S}}$ and this proves the theorem. $\square$

We prove Claims (1) and (2).

Proof of Claim (1). Recall that there are three conditions to check.

(a) $\bigcap_{n\in\mathbb{Z}} \Gamma_n = 0$. Let $M_{-\infty} = \bigcap_{n\in\mathbb{Z}} \Gamma_n$. It is easy to check that $M_{-\infty}$ is an A-sub-bimodule of M. We see that $\mathrm{gr}^{\Gamma}(M_{-\infty}) = 0$: for all $i \in \mathbb{Z}$, $(\Gamma_i \cap M_{-\infty})/(\Gamma_{i-1} \cap M_{-\infty}) = (\Gamma_{i-1} \cap M_{-\infty})/(\Gamma_{i-1} \cap M_{-\infty}) = 0$. Therefore the map $\mathrm{gr}^{\mathcal{F}}(M_{-\infty}) \to \mathrm{gr}^{\Gamma}(M_{-\infty})$ given by the restriction of the map α above, is the zero map. It follows from Proposition 3.13 that $\mathrm{gr}^{\mathcal{F}}(M_{-\infty}) \subseteq K$ where K is the kernel of β. By Lemma 3.12

$$\mathrm{supp}\ \mathrm{gr}^{\mathcal{F}}(M_{-\infty}) \cap U = \emptyset. \tag{3.5}$$

Now since $M_{-\infty}$ is an A-sub-bimodule of A/I, there is an ideal J of A such that $I \subseteq J \subseteq A$ and $M_{-\infty} = J/I$. Suppose that $M_{-\infty} \neq 0$. Then J properly contains I. It is a consequence of [49, Propositions 3.15 and 6.6] that $\mathrm{Dim}\mathcal{V}(\mathrm{gr}^{\mathcal{F}} J) < \mathrm{Dim}\mathcal{V}(\mathrm{gr}^{\mathcal{F}} I) = \mathrm{Dim}\bar{\mathcal{S}}$. Let $\mathfrak{p}$ be the defining ideal of $\bar{\mathcal{S}}$. The equality of closed sets (where the support is considered over $\mathrm{gr}^{\mathcal{F}} A$)

$$\mathrm{supp}\ \mathrm{gr}^{\mathcal{F}}(A/I) = \mathrm{supp}\ \mathrm{gr}^{\mathcal{F}}(J/I) \cup \mathrm{supp}\ \mathrm{gr}^{\mathcal{F}}(A/J)$$

implies that $\mathfrak{p} \in$ supp $\mathrm{gr}^{\mathcal{F}}(J/I)$ and therefore that $\bar{\mathcal{S}} \subseteq$ supp $\mathrm{gr}^{\mathcal{F}}(J/I)$. Hence $\mathcal{S} \subseteq \bar{\mathcal{S}} \subseteq$ supp $\mathrm{gr}^{\mathcal{F}}(J/I) =$ supp $\mathrm{gr}^{\mathcal{F}}(M_{-\infty})$. This contradicts (3.5) and so $M_{-\infty} = 0$.

(b) $\bigcup_{n\in\mathbb{Z}} \Gamma_n = M$. This is straightforward because $\mathcal{F}_n \subseteq \Gamma_n$ implies $M = \bigcup_n \mathcal{F}_n \subseteq \bigcup_n \Gamma_n \subseteq M$.

(c) $\mathrm{gr}^{\Gamma} M$ *is a finitely generated* $\mathrm{gr}^{\mathcal{F}} A$ *– module.* To prove this we in fact show that $\mathcal{M}(U)$ is a finitely generated $\mathrm{gr}^{\mathcal{F}}A$-module (which proves (c) by Proposition 3.13, since $\mathrm{gr}^{\mathcal{F}}A$ is Noetherian). We would like to show that $\mathcal{M}(U)$ is finitely generated and so by Theorem 3.14 it suffices to show that each prime $P \in U \cap \mathrm{Ass}\ \mathrm{gr}^{\mathcal{F}}M$ satisfies (3.4) with $R = \mathrm{gr}^{\mathcal{F}}A$. Let $P \in U \cap \mathrm{Ass}\ \mathrm{gr}^{\mathcal{F}}M \subseteq U \cap \mathrm{supp}\ \mathrm{gr}^{\mathcal{F}}M = \mathcal{S}$. By Lemma 1.29, P is a Poisson prime ideal of $\mathrm{gr}^{\mathcal{F}}A$. Now $U \cap \overline{P}$ is a nonempty open subset of $\overline{P}$ which means that it contains a closed point $\mathfrak{m}$, and $U \cap \overline{P} \subseteq \mathcal{S}$ implies that $\mathfrak{m}$ has rank $\mathrm{Dim}\, Y$. We conclude that $\mathrm{Dim}\, \overline{P} = \mathrm{Dim}\, Y$ by [11, Lemma 3.1(5)], and therefore $\overline{P} = \bar{\mathcal{S}}$. Thus condition (3.4) is a consequence of Lemma 3.15 (2) and $\mathcal{M}(U)$ is finitely generated. This proves (c) and concludes the proof of (1). □

Proof of Claim (2). We must show that $P \notin \bar{\mathcal{S}} \Rightarrow \mathcal{M}(U)_P = 0$. Let $P \notin \bar{\mathcal{S}}$. This is equivalent to there being a neighbourhood, Y, of P in X such that $Y \cap \mathcal{S} = \emptyset$. Without loss of generality, we may assume that Y is some standard open set O_g for some $g \in \mathrm{gr}^{\mathcal{F}}A$ with $g \notin P$. We have $U = X \setminus \mathcal{V}(f_1, \ldots, f_t) = O_{f_1} \cup \cdots \cup O_{f_t}$; for each i, $O_{f_i g} = O_{f_i} \cap O_g$ is an open subset of U which intersects $\mathcal{S}$ trivially. Since $U \cap Y = \mathcal{S}$ we conclude that $O_{f_i g}$ is contained in the open set $X \setminus Y$, and therefore that $Y \subseteq \mathcal{V}(f_i g)$. By considering the ideals of these subvarieties we deduce that $\mathrm{rad}\langle f_i g\rangle \subseteq \mathrm{rad}(\mathrm{ann}\ \mathrm{gr}^{\mathcal{F}}M)$. Hence there are integers k_i so that $(f_i g)^{k_i} \in \mathrm{ann}\ \mathrm{gr}^{\mathcal{F}}M$. We consider a typical element $(\frac{m_1}{f_1{}^{N_1}}, \ldots, \frac{m_t}{f_t{}^{N_t}}) \in \mathcal{M}(U)$. Let $k = max_i\{k_i\}$, then for all i:

$$(f_i g)^k m_i = 0 \Rightarrow \frac{g^k m_i}{f_i{}^{N_i}} = 0 \in \mathrm{gr}^{\mathcal{F}} M_{f_i} \Rightarrow \mathcal{M}(U)_P = 0$$

and this proves (2). □

3.5 Remarks

1. One can easily see that the map Ω from (3.1) is neither injective nor surjective in general. The fact it is not always surjective follows from Proposition 1.25. Here the varieties X_c are all smooth for $c \neq 0$ and so have just one leaf, Corollary 1.45, whereas X_0 is singular so has more than one leaf. On the other hand, the failure of injectivity follows directly from the

calculation of the varieties X_c associated to the cyclic subgroups $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$, which is found in [8]. More precisely, X_0 has two leaves in this case and there exist X_c with > 2 leaves when $|\Gamma| \geq 4$.

Chapter 4

The symplectic leaves of X_c via quivers

We use Theorem 4.2 to describe the symplectic leaves of a Marsden-Weinstein reduction for which the local normal form exists. We apply this to the reductions associated to representations of quivers to deduce that the stratification by symplectic leaves equals the stratification by representation type. Finally we extend Theorem 2.10 to all values of the parameter c to establish an isomorphism between the variety X_c for wreath products and a Marsden-Weinstein reduction for a quiver. Furthermore we show that this isomorphism is Poisson up to nonzero scalar multiple and so identifies the symplectic leaves of each variety. We discuss symplectic leaves for reductions in Section 4.1. The proof of our main result, Corollary 4.25, occupies Sections 4.2 and 4.3.

4.1 The symplectic leaves of Marsden-Weinstein reductions

Throughout this section let M be a symplectic variety with corresponding 2-form, ω. Let G be a connected reductive algebraic group with a Hamiltonian action on M and denote by μ the resulting moment map. Let $\mathcal{L}$ be a fixed point in $\mathfrak{g}^*$. Let $Z := \mu^{-1}(\mathcal{L})$ and let $M_{\mathcal{L}}$ denote the corresponding Marsden-Weinstein reduction, $Z//G$. Let $\pi : Z \to M_{\mathcal{L}}$ be the orbit map. Recall from Section 2.1 that we can stratify $M_{\mathcal{L}}$ by orbit type:

$$M_{\mathcal{L}} = \bigcup_{\tau \in \mathcal{T}} (M_{\mathcal{L}})_\tau \tag{4.1}$$

where $\mathcal{T}$ is the set of conjugacy classes of subgroups of G which occur as the stabilisers of points in Z.

We would like to relate this stratification to the stratification by symplectic leaves. Since symplectic leaves are by definition connected, we take connected components of the orbit type strata. For any $\tau \in \mathcal{T}$ we can decompose $(M_{\mathcal{L}})_\tau$ into its connected components C^i_τ where i runs over some index set $\mathcal{I}_\tau$. Since $(M_{\mathcal{L}})_\tau$ is locally closed in the Zariski topology (Proposition 2.3), the C^i_τ are subvarieties of $(M_{\mathcal{L}})_\tau$ and the set $\mathcal{I}_\tau$ is finite. Let $Z_{(H)} = \{z \in Z : G_z \text{ is conjugate to } H\}$.

Lemma 4.1. *Let $\tau \in \mathcal{T}$ and let $H \in \tau$. Then $\pi^{-1}((M_{\mathcal{L}})_\tau) \cap Z_{(H)}$ is a nonempty open G-stable subset of $Z_{(H)}$.*

Proof. Let $z \in Z_{(H)}$. By (4.1) there exists a $\sigma \in \mathcal{T}$ such that $\pi(z) \in (M_{\mathcal{L}})_\sigma$. Let $X_\sigma = \bigcup_{\nu \leq \sigma} (M_{\mathcal{L}})_\nu$ which is a closed subset of $M_{\mathcal{L}}$ by Proposition 2.3. If the orbit $G \cdot z$ is closed in Z then $\sigma = \tau$. If this orbit is not closed then $\sigma \neq \tau$ and the unique closed orbit in $\pi^{-1}(\pi(z))$ has dimension strictly less than $\operatorname{Dim} G \cdot z$, therefore $\tau \not\leq \sigma$ and we have $(M_{\mathcal{L}})_\tau \cap X_\sigma = \emptyset$. Let $\mathcal{S} = \{\sigma \in \mathcal{T} : \pi(z) \in (M_{\mathcal{L}})_\sigma \text{ for some } z \in Z_{(H)}\}$ and let $\mathcal{S}' = \mathcal{S} \backslash \{\tau\}$. By definition we have $\pi^{-1}(\bigcup_{\sigma \in \mathcal{S}} X_\sigma) \cap Z_{(H)} = Z_{(H)}$. Furthermore we have $\bigcup_{\sigma \in \mathcal{S}} X_\sigma = (M_{\mathcal{L}})_\tau \sqcup \bigcup_{\sigma \in \mathcal{S}'} X_\sigma$. Therefore

$$\pi^{-1}((M_{\mathcal{L}})_\tau) \cap Z_{(H)} = (\pi^{-1}(\bigcup_{\sigma \in \mathcal{S}} X_\sigma) \cap Z_{(H)}) \setminus (\pi^{-1}(\bigcup_{\sigma \in \mathcal{S}'} X_\sigma) \cap Z_{(H)}) = Z_{(H)} \setminus (\pi^{-1}(\bigcup_{\sigma \in \mathcal{S}'} X_\sigma) \cap Z_{(H)})$$

is open in $Z_{(H)}$. That this set is nonempty and G-stable follows from the definition of $Z_{(H)}$ and Lemma 2.1. □

The following theorem is based very closely on [74, Theorem 2.1]. This earlier result was proved over $\mathbb{R}$ and for a compact Lie group, we provide full details to verify that the result carries over to our situation.

Theorem 4.2. *Suppose that the local normal form for μ exists. Then the decomposition $M_{\mathcal{L}} = \bigsqcup \{C^i_\tau : \tau \in \mathcal{T},\ i \in \mathcal{I}_\tau\}$ is a stratification of $M_{\mathcal{L}}$ into a disjoint union of symplectic manifolds. Denote by $(\omega_0)^i_\tau$ the symplectic form on C^i_τ. The pullback of the symplectic form, $\pi^*(\omega_0)^i_\tau$, to $Z^i_{(H)} := Z_{(H)} \cap \pi^{-1}(C^i_\tau)$ equals the restriction to $Z^i_{(H)}$ of the symplectic form ω.*

Proof. We first give the proof when $\mathcal{L} = \mathbf{0}$. Let $\tau \in \mathcal{T}$ and choose a representative of τ, H. Let $\zeta \in (M_{\mathcal{L}})_\tau$ and $Gm \subseteq Z$ be the unique closed orbit in $\pi^{-1}(\zeta)$. Let $\Upsilon = (G \times \mathfrak{h}^{\perp *} \times \hat{M})//H$. Since we are assuming that the local normal form for μ exists we can work in the model space Υ;

that is, there is G-invariant holomorphic bijection, intertwining symplectic structures, between a neighbourhood of Gm in M and a neighbourhood of $G/H \times 0 \times 0$ in Υ. Recall that the moment map for the action of G on Υ is denoted J.

Let $\Upsilon_{(H)} = \{y \in \Upsilon : G_y$ is conjugate to $H\}$, where G_y denotes the stabiliser of y with respect to the action of G on Υ. Let $y \in \Upsilon_{(H)}$. Then we can write $y = [g, \xi, \hat{m}]$, and for all $x \in G_y$, $[xg, \xi, \hat{m}] = [g, \xi, \hat{m}]$, that is, there exists $h \in H$ such that $(xg, \xi, \hat{m}) = (gh, \mathrm{Ad}^*(h^{-1})\xi, h^{-1}\hat{m})$. By considering the first component of y we deduce that $g^{-1}G_y g = H$, and therefore for all $h \in H$, $\mathrm{Ad}^*(h)\xi = \xi$ and $h\hat{m} = \hat{m}$. The moment map $\hat{\mu}$ is obtained from the Hamiltonian function described in Theorem 2.18. Thus $h\hat{m} = \hat{m}$ for all $h \in H$ implies that $\hat{\mu}(\hat{m}) = 0$. Therefore

$$\begin{aligned} J^{-1}(0) \cap \Upsilon_{(H)} &= \{[g, \xi, \hat{m}] : \mathrm{Ad}^*(h)\xi = \xi, h\hat{m} = \hat{m} \text{ for all } h \in H \text{ and } \xi + \hat{\mu}(\hat{m}) = 0\} \\ &= \{[g, \xi, \hat{m}] : \xi = 0 \text{ and } h\hat{m} = \hat{m} \text{ for all } h \in H\} \\ &= (G \times \hat{M}_H)//H = G/H \times \hat{M}_H, \end{aligned}$$

where $\hat{M}_H$ is the subspace of fixed points of $\hat{M}$, which is a symplectic subspace of $\hat{M}$.

Let C^i_τ be the connected component of $(M_{\mathcal{L}})_\tau$ containing ζ. By Lemma 4.1 $Z^i_{(H)}$ is a nonempty open G-stable subset of $Z_{(H)}$. Thus the local normal form allows us to identify a neighbourhood of Gm in $Z^i_{(H)}$ with $J^{-1}(0) \cap \Upsilon_{(H)}$ in a G-equivariant way. Considering the orbit maps restricted to each of these we see that there is a (complex) neighbourhood of $\zeta \in C^i_\tau$ which is holomorphically bijective with an open subset of $(G/H \times \hat{M}_H)//G = \hat{M}_H$. Therefore C^i_τ a symplectic manifold.

Let ω_H be the symplectic form on $\hat{M}_H$, and consider the G-quotient map, $\sigma : G/H \times \hat{M} \to \hat{M}$. If we think of the former space as a submanifold of the symplectic manifold Υ, whose symplectic form we denote by ϖ, then it is clear that $\varpi|_{G/H \times \hat{M}} = \sigma^*\omega_H$. It follows from the local normal form that the quotient map $\pi : Z^i_{(H)} \to C^i_\tau$ can be taken to be, locally at least, the quotient map σ. Therefore $\omega|_{Z^i_{(H)}} = \pi^*(\omega_0)^i_\tau$.

For general $\mathcal{L}$ we use the 'shifting trick' to reduce to the case above. $\mathcal{L}$ is a symplectic variety in a trivial way, so we form the product $M \times -\mathcal{L}$, which is a symplectic variety with moment map $\mu'(m, -\mathcal{L}) = \mu(m) + -\mathcal{L}$. The local normal form for μ' will exist because it does so for μ, and we can identify $\mu'^{-1}(\mathbf{0})//G$ with $\mu^{-1}(\mathcal{L})//G$. $\square$

We note that the strata $(M_{\mathcal{L}})_\tau$ are not necessarily connected and it is possible that the connected components have differing dimensions (see [74, Remark 2.6]). However, by Proposition 2.4 when

M is a finite dimensional vector space on which G acts linearly then the $(M_{\mathcal{L}})_\tau$ are all irreducible in the Zariski topology. In particular they are connected.

One can explain the phenomenon of different components of $(M_{\mathcal{L}})_\tau$ being symplectic manifolds of possibly different dimensions in terms of the G-action on M. Let Gx, Gy be closed orbits in M such that their stabilisers are conjugate to $H \subseteq G$. The vector spaces $\hat{M}_x$ and $\hat{M}_y$ are both H-modules but it is possible that $(\hat{M}_x)_H$ and $(\hat{M}_y)_H$ have different dimensions. This notion is put into a general framework in [54].

In light of the theorem, we know that the $Z^i_{(H)}$ are manifolds and it turns out that these are well behaved with respect to Hamiltonian vector fields of invariant functions.

Lemma 4.3. *Let $f \in \mathcal{O}(M)^G$, and let Ξ_f be the Hamiltonian vector field of f. Then for all $z \in Z^i_{(H)}$, $\Xi_f(z)$ is contained in the tangent space $T_z Z^i_{(H)}$.*

Proof. It follows from [55, Proposition 10.5.2] that f being G-invariant means that travelling along the integral curve to $\Xi_f(z)$ preserves stabiliser type. Thus the integral curve to $\Xi_f(z)$ is contained in $Z_{(H)}$. □

Theorem 4.2 allows us to define a bracket on $\mathcal{O}(M_{\mathcal{L}})$: we describe this and compare it to the Poisson bracket from Proposition 2.20. Let $\{-,-\}$ be the canonical Poisson bracket on $\mathcal{O}(M)$ and $\{-,-\}_1$ the bracket on $\mathcal{O}(M_{\mathcal{L}})$ induced from $\{-,-\}$ as in Proposition 2.20. We can define another bracket, $\{-,-\}_2$, on $\mathcal{O}(M_{\mathcal{L}})$ in the following way. Let $f, g \in \mathcal{O}(M_{\mathcal{L}})$ and $p \in C^i_\tau$ for some $\tau \in \mathcal{T}$ and some $i \in \mathcal{I}_\tau$. Then define $\{f,g\}_2(p) = \{f\big|_{\tau,i}, g\big|_{\tau,i}\}^i_\tau(p)$ where $f\big|_{\tau,i}, g\big|_{\tau,i}$ denote the restrictions to C^i_τ of f and g respectively and $\{-,-\}^i_\tau$ is the Poisson bracket on $\mathcal{O}^{hol}_{C^i_\tau}(C^i_\tau)$ induced by the symplectic form, $(\omega_0)^i_{(\tau)}$. It is not clear that this is a Poisson bracket on $\mathcal{O}(M_{\mathcal{L}})$; we remedy this below.

The following proposition is based on [74, Proposition 3.1].

Proposition 4.4. *The brackets $\{-,-\}_1$ and $\{-,-\}_2$ are equal.*

Proof. Let $f, g \in \mathcal{O}(M_{\mathcal{L}})$. Let $\tau \in \mathcal{T}$, $H \in \tau$ and $p_0 \in C^i_\tau$ for some $i \in \mathcal{I}_\tau$. It suffices to show that $\{f,g\}_1(p_0) = \{f,g\}_2(p_0)$. Let $p \in \pi^{-1}(p_0)$ and let $\tilde{f}, \tilde{g} \in \mathcal{O}(M)^G$ be such that $\tilde{f} + I^G = f$ and $\tilde{g} + I^G = g$ where I is the ideal of functions in $\mathcal{O}(M)$ vanishing on $\mu^{-1}(\mathcal{L})$. Then $\{f,g\}_1(p_0) = \{\tilde{f}, \tilde{g}\}(p)$.

By Lemma 4.3, $\Xi_{\tilde{f}}$, the Hamiltonian vector field of $\tilde{f}$ with respect to the symplectic form on M, is tangent to $Z^i_{(H)}$. In particular we have $\Xi_{\tilde{f}}(p) \in T_p(Z^i_{(H)})$.

By Theorem 4.2, we have $\omega\big|_{Z^i_{(H)}} = \pi^*(\omega_0)^i_{(\tau)}$. It is shown in the proof of Theorem 4.2 that the map $\pi : Z^i_{(H)} \to C^i_\tau$ is a (complex analytic) fibration of type G/H. Thus the differential $d\pi_p : T_p(Z^i_{(H)}) \to T_{p_0}(C^i_\tau)$ is surjective. For all $\phi \in T_p(Z^i_{(H)})$,

$$(d\pi_p(\phi))([f]) = \phi([\tilde{f}]) = \omega(\Xi_{\tilde{f}}(p), \phi) = \pi^*(\omega_0)^i_{(\tau)}(\Xi_{\tilde{f}}(p), \phi) = (\omega_0)^i_{(\tau)}(d\pi_p(\Xi_{\tilde{f}}(p)), d\pi_p\phi).$$

Therefore $d\pi_p(\Xi_{\tilde{f}}(p)) = \Xi_f(p_0)$ (the latter Hamiltonian vector field being defined with respect to $(\omega_0)^i_{(\tau)}$). By definition of the bracket $\{-,-\}_2$ we have $\{f,g\}_2(p_0) = (\omega_0)^i_{(\tau)}(\Xi_f(p_0), \Xi_g(p_0))$, and then one calculates

$$\begin{aligned}\{f,g\}_2(p_0) &= (\omega_0)^i_{(\tau)}(\Xi_f(p_0), \Xi_g(p_0)) = (\omega_0)^i_{(\tau)}(d\pi_p(\Xi_{\tilde{f}}(p)), d\pi_p(\Xi_{\tilde{g}}(p))) \\ &= \pi^*(\omega_0)^i_{(\tau)}(\Xi_{\tilde{f}}(p), \Xi_{\tilde{g}}(p)) = \omega(\Xi_{\tilde{f}}(p), \Xi_{\tilde{g}}(p)) = \{\tilde{f}, \tilde{g}\}(p) = \{f,g\}_1(p_0).\end{aligned}$$

□

Thus we refer to the Poisson bracket on $\mathcal{O}(M_{\mathcal{L}})$ by $\{-,-\}$. The next proposition is a generalisation of Example 1.37. Recall that in that situation we have M a symplectic vector space and G a finite subgroup of the symplectic group of M. Then the symplectic leaves of M/G are the orbit type strata.

Proposition 4.5. *The C^i_τ are the symplectic leaves of $M_{\mathcal{L}}$. There are finitely many symplectic leaves and they are irreducible locally closed subvarieties.*

Proof. Let $\tau \in \mathcal{T}$. The set $X = \bigcup_{\nu \leq \tau} \bigcup_{i \in I_\nu} C^i_\nu$ is closed by Proposition 2.3. Let I be the defining ideal of X. Let $p \in X$ so that $p \in C^j_\nu$ for some ν and j. For any $f \in \mathcal{O}(M_{\mathcal{L}})$ and $i \in I$ we have $\{f, i\}(p) = \{f\big|_{\nu,j}, i\big|_{\nu,j}\}^j_\nu(p) = \{f\big|_{\nu,j}, 0\}^j_\nu(p) = 0$, that is I is a Poisson ideal of $\mathcal{O}(M_{\mathcal{L}})$. Thus X is a Poisson subvariety of $M_{\mathcal{L}}$.

Let $p \in C^i_\tau$. Let $\mathcal{S}$ be the symplectic leaf through p in $M_{\mathcal{L}}$. Since X is Poisson we have $\mathcal{S} \subseteq X$ by Lemma 1.41. Proposition 4.4 says that the restriction map $\mathcal{O}(X) \to \mathcal{O}^{hol}_{C^i_\tau}(C^i_\tau)$ is Poisson. By Proposition 2.3 C^i_τ is open in X and therefore $C^i_\tau \subseteq \mathcal{S}$ by Proposition 1.42.

On the other hand, the set $X' = X \setminus C^i_\tau$ is closed and an identical argument to that of the first paragraph shows that it is a Poisson subvariety of $M_{\mathcal{L}}$. Now if $\mathcal{S} \cap X' \neq \emptyset$ then $\mathcal{S} \subseteq X'$ by Lemma

1.41, but this cannot happen because $p \notin X'$. Therefore $\mathcal{S} \subseteq X \setminus X' = C^i_\tau$. Thus the C^i_τ are the symplectic leaves and the remaining facts follow from Proposition 2.3 and Theorem 1.43 (ii). □

We have given two examples of Marsden-Weinstein reduction: Calogero-Moser space and the spaces $\mathcal{N}(\lambda, \alpha)$ associated to any quiver. In order to apply the above proposition we require that the local normal form of the moment map exists and that the reduction occurs at a coadjoint orbit which is a fixed point. These are true in the example of representations of deformed preprojective algebras by Corollary 2.36.

Corollary 4.6. *The symplectic leaves of $\mathcal{N}(\lambda, \alpha)$ are the representation type strata.*

Proof. By Lemma 2.26 the representation type strata are equal to the orbit type strata, and by Proposition 2.4 these strata are connected. □

We can use this corollary and the facts we know about symplectic leaves to calculate the smooth locus of $\mathcal{N}(\lambda, \alpha)$. We emphasise that this result is already known, as follows. For $\alpha \in \Sigma_\lambda$, Le Bruyn proved that the smooth locus of $\mathcal{N}(\lambda, \alpha)$ is the stratum of representation type $(1, \alpha)$, [52, Theorem 3.2]. For general α we have a decomposition $\alpha = \sigma^{(1)} + \cdots \sigma^{(r)}$ with $\sigma^{(i)} \in \Sigma_\lambda$ as in Theorem 2.30 (1). Then collecting terms we have $\mathcal{N}(\lambda, \alpha) \cong \prod_{i=1}^s \operatorname{Sym}^{m_i} \mathcal{N}(\lambda, \sigma^{(i)})$ for some s. One observes that for any smooth affine algebraic variety X with $\operatorname{Dim} X \geq 2$ the smooth locus of $\operatorname{Sym}^n X$ is the set $\{[x_1, \ldots, x_n] : x_i \neq x_j \text{ for all } i \neq j\}$. It follows that the smooth locus of $\mathcal{N}(\lambda, \alpha)$ is the stratum of representation type $(1, \sigma^{(1)}; \ldots; 1, \sigma^{(r)})$.

We give a proof utilising Corollary 4.6.

Theorem 4.7. *Let $\lambda \in \mathbb{C}^I$ and $\alpha \in \mathbb{N}R^+_\lambda$. The smooth locus of $\mathcal{N}(\lambda, \alpha)$ coincides with the stratum of representation type $(1, \sigma^{(1)}; \ldots; 1, \sigma^{(r)})$, where $\alpha = \sigma^{(1)} + \cdots \sigma^{(r)}$ is the unique decomposition from Theorem 2.30 (1).*

Proof. Let $\tau = (1, \sigma^{(1)}; \ldots; 1, \sigma^{(r)})$. Then $\operatorname{Dim}\mathcal{R}_\tau = \operatorname{Dim}\mathcal{N}(\lambda, \alpha)$ by Proposition 2.28 and Theorem 2.30 (3). Therefore $\overline{\mathcal{R}_\tau} = \mathcal{N}(\lambda, \alpha)$ because both varieties are irreducible. By Lemma 4.6 and Theorem 1.43 (ii), $\mathcal{R}_\tau$ is the smooth locus of $\mathcal{N}(\lambda, \alpha)$. □

In particular, we have established that the smooth locus of $\mathcal{N}(\lambda, \alpha)$ is always symplectic. We can determine whether $\mathcal{N}(\lambda, \alpha)$ is smooth in terms of roots of Q. We write $\alpha = \sigma^{(1)} + \cdots \sigma^{(r)} = \sum_{i=1}^s m_i \sigma^{(i)}$, by collecting like terms.

Corollary 4.8. *$\mathcal{N}(\lambda,\alpha)$ is smooth if and only if $\alpha=\sum_{i=1}^{s} m_i\sigma^{(i)}$ is the only possible decomposition of α as a sum of elements of Σ_λ and for each i, $p(\sigma^{(i)})>0$ implies that $m_i=1$.*

Proof. By the theorem, $\mathcal{N}(\lambda,\alpha)$ is smooth if and only if $(1,\sigma^{(1)};\ldots;1,\sigma^{(r)})$ is the unique representation type. The result follows from Lemma 2.27. □

We also show how to work out the symplectic leaves of $\mathcal{N}(\lambda,\alpha)$. This amounts to finding each of the possible decompositions of α as a sum of vectors in Σ_λ and listing the representation types arising out of each decomposition.

For a decomposition

$$\alpha=m_1\alpha_1+\cdots+m_s\alpha_s+n_1\beta_1+\cdots+m_t\beta_t \tag{4.2}$$

where $\alpha_i,\beta_j\in\Sigma_\lambda$ for all i, $p(\alpha_i)=0$ and $p(\beta_j)>0$ for all j, the corresponding representation types of $\mathcal{N}(\lambda,\alpha)$ are labeled by t-tuples of partitions. More precisely, let P_j be the set of partitions of n_j, and for any $\sigma\in P_i$ denote its length by $l(\sigma)$. All the representation types coming from (4.2) are parametrised naturally by $P_1\times\cdots\times P_t$ by Lemma 2.27. For each $(\sigma_1,\ldots,\sigma_t)\in P_1\times\cdots\times P_t$ we denote the corresponding representation type by $\tau_{(\sigma_1,\ldots,\sigma_t)}$. Then $\mathrm{Dim}\mathcal{R}_{\tau_{(\sigma_1,\ldots,\sigma_t)}}=2\sum_{j=1}^{t}l(\sigma_j)p(\beta_j)$ by Proposition 2.28.

4.2 An isomorphism between $\mathcal{N}(\lambda,\alpha)$ and $Cal_c//\,\mathrm{G}(n\delta)$

We recall some notation introduced in Example 2.9. Let n be an integer greater than one. Let L be a two dimensional symplectic vector space with symplectic basis x,y. Let $\Gamma\subset SL(L)$ be a finite subgroup. Let R be the regular representation of Γ. Denote by $e_\Gamma\in\mathrm{End}_\Gamma(R)$ the projector onto the trivial representation. Let $\underline{c}:\Gamma\setminus\{1\}\to\mathbb{C}$ be a class function. Let $\mathbf{c}\in\mathrm{End}_\Gamma(R)$ multiplication by the central element $\sum_{\Gamma\setminus\{1\}}\underline{c}(\gamma)\gamma$. Let $\mathcal{O}$ be the $\mathrm{GL}(n,\mathbb{C})$-conjugacy class formed by all $n\times n$-matrices of the form $P-\mathrm{Id}$, where P is a semisimple rank one matrix such that $\mathrm{tr}(P)=\mathrm{tr}(\mathrm{Id})=n$. Define

$$Cal_c=\{\nabla\in\mathrm{Hom}_\Gamma(L,\mathrm{End}_\mathbb{C}(R^n)):[\nabla(x),\nabla(y)]\in\frac{1}{2}c_1|\Gamma|\mathcal{O}\otimes e_\Gamma+\mathrm{Id}\otimes\mathbf{c}\},$$

where $c_1\in\mathbb{C}$ and $\frac{1}{2}c_1|\Gamma|\mathcal{O}\otimes e_\Gamma+\mathrm{Id}\otimes\mathbf{c}\subseteq\mathrm{End}(\mathbb{C}^n)\otimes\mathrm{End}_\Gamma(R)=\mathrm{End}_\Gamma(\mathbb{C}^n\otimes R)=\mathrm{End}_\Gamma(R^n)$.

Recall from Example 2.11 that for $\alpha\in\mathbb{N}^k$ we define

$$\hat{\mathrm{G}}(\alpha)=\prod_{i=1}^{k}\mathrm{GL}(\alpha_i,\mathbb{C}).$$

For any α we have $\mathbb{C}^\times \subseteq \hat{\mathrm{G}}(\alpha)$ as the subgroup $\{(\lambda \mathrm{Id}_{\alpha_i}) : \lambda \in \mathbb{C}^\times\}$. If $S_0, \ldots, S_k$ are the irreducible representations of Γ (with S_0 the trivial representation) and $\delta \in \mathbb{N}^{k+1}$ the vector with $\delta_i = \mathrm{Dim}_{\mathbb{C}} S_i$ then there is an isomorphism of groups $\hat{\mathrm{G}}(n\delta) \cong \mathrm{Aut}_\Gamma(R^n)$. Thus $\mathrm{G}(n\delta) := \hat{\mathrm{G}}(n\delta)//\mathbb{C}^\times$ acts on Cal_c and the quotient $Cal_c// \mathrm{G}(n\delta)$ is Calogero-Moser space for Γ_n.

We make the connection between Calogero-Moser space for Γ_n and the representations of certain deformed preprojective algebras. Recall the McKay graph of Γ from (1.4): it is simply laced and so in particular contains no double edges. Let Q be the extended Dynkin diagram given by choosing an orientation of the McKay graph. It has vertex set I where each vertex corresponds to an irreducible representation of Γ so we have $I = \{0, \ldots, k\}$. Then δ is the minimal imaginary root for Q as in Section 2.3. Define a linear map

$$\begin{aligned}\lambda : \mathbb{C}^{k+1} \longrightarrow \mathbb{C}^I, \quad c = (c_1, \underline{c}) \mapsto \lambda(c_1, \underline{c})_k =& (-\frac{1}{2}c_1 \mathrm{tr}_{S_k} \sum_{\gamma \in \Gamma} \gamma + \mathrm{tr}_{S_k} \sum_{\gamma \in \Gamma \setminus \{1\}} \underline{c}(\gamma)\gamma) \\ =& (-\frac{1}{2}c_1 |\Gamma| n\delta_{0k} + \mathrm{tr}_{S_k}\mathbf{c}).\end{aligned}$$

Now consider the quiver Q' defined by adding a vertex ∞ to the quiver Q and adding one arrow from the vertex ∞ to the vertex 0. Let $I' = \{\infty\} \cup I$ and let $\lambda'(c) = (-\lambda(c) \cdot n\delta, \lambda(c)) \in \mathbb{C}^{I'}$. Let $\epsilon_\infty + n\delta \in \mathbb{C}^{I'}$ be the vector with 1 in the ∞ position and $n\delta_i$ in the ith position for $i \in I$. We have the Marsden-Weinstein reduction corresponding to Q', $\lambda'(c)$ and $\epsilon_\infty + n\delta$:

$$\begin{aligned}\mathcal{N}\left(\lambda'(c), \epsilon_\infty + n\delta\right) = \{(B, i, j) \in& \bigoplus_{a \in \overline{A}} \mathrm{Mat}(n\delta_{h(a)} \times n\delta_{t(a)}, \mathbb{C}) \oplus \mathrm{Mat}(n\delta_0 \times 1, \mathbb{C}) \\ &\oplus \mathrm{Mat}(1 \times n\delta_0, \mathbb{C}) : \mu_{n\delta}(x) + ij - ji = \lambda'(c)\}// \mathrm{G}(\epsilon_\infty + n\delta).\end{aligned}$$

Remark 4.9. As noted in [16, Section 1] $\mathcal{N}\left(\lambda'(c), \epsilon_\infty + n\delta\right)$ can be described as one of the quiver varieties defined by Nakajima, see [61].

There is an isomorphism $\hat{\mathrm{G}}(n\delta) \cong \mathrm{G}(\epsilon_\infty + n\delta)$ given by $(g_i) \mapsto (1, (g_i))\mathbb{C}^*$. We shall identify these two groups in what follows.

Theorem 4.10. *For any Γ, $n > 1$ and $c \in \mathbb{C}^{k+1}$ there is a Poisson isomorphism*

$$\mathcal{N}\left(\lambda'(c), \epsilon_\infty + n\delta\right) \cong Cal_c// \mathrm{G}(n\delta).$$

We prove Theorem 4.10 in several stages. We begin by noting an equivalent form for $Cal_c// \mathrm{G}(n\delta)$. By the adjunction of Hom and $\otimes$ we have $\mathrm{Hom}_\Gamma(L, \mathrm{End}_{\mathbb{C}}(R^n)) = \mathrm{Hom}_\Gamma(L \otimes R^n, R^n)$, and this identification is $\mathrm{G}(n\delta)$-equivariant. Let $\nabla \in \mathrm{Hom}_\Gamma(L \otimes R^n, R^n)$ so that $\nabla = x^* \otimes \phi + y^* \otimes \psi$ for some

$\phi, \psi \in \mathrm{End}_{\mathbb{C}}(R^n)$. Then $[\nabla(x), \nabla(y)] = \phi\psi - \psi\phi$. In this way it is straightforward to check that $[\nabla(x), \nabla(y)]$ is the map

$$[\nabla, \nabla] : R^n \xrightarrow{\zeta \otimes Id} L \otimes L \otimes R^n \xrightarrow{Id \otimes \nabla} L \otimes R^n \xrightarrow{\nabla} R^n \tag{4.3}$$

where ζ is the Γ-map

$$\zeta : \mathbb{C} \to L \otimes L, \ \zeta(1) = x \otimes y - y \otimes x. \tag{4.4}$$

Therefore

$$Cal_c // \mathrm{G}(n\delta) \cong \{\nabla \in \mathrm{Hom}_\Gamma(L \otimes R^n, R^n) : [\nabla, \nabla] \in \frac{1}{2} c_1 |\Gamma| \mathcal{O} \otimes e_\Gamma + \mathrm{Id} \otimes \mathbf{c}\} // \mathrm{G}(n\delta).$$

We shall apply the shifting trick (used in a trivial way in the proof of Theorem 4.2) to $Cal_c // \mathrm{G}(n\delta)$. Let $m = -\frac{1}{2} n c_1 |\Gamma|$. Let $\mathcal{L}_m$ be the $\mathrm{GL}(n, \mathbb{C})$-conjugacy class formed by all $n \times n$-matrices which are semisimple, whose rank is less than or equal to one and whose trace is equal to m (this is the same notation used in Example 2.22). We think of $\mathrm{End}(n\delta)_0$ as a subspace of $\mathrm{End}(n\delta)$ and the latter is acted upon by $\mathrm{G}(n\delta)$. It is important to make this observation for what follows because $-\frac{1}{2} c_1 |\Gamma| \mathrm{Id} \otimes e_\Gamma \in \mathrm{End}(n\delta)$ but not in $\mathrm{End}(n\delta)_0$. Let

$$Y = \{(\nabla, P) \in \mathrm{Hom}_\Gamma(L \otimes R^n, R^n) \times \mathcal{L}_m \otimes e_\Gamma : [\nabla, \nabla] + P = -\frac{1}{2} c_1 |\Gamma| \mathrm{Id} \otimes e_\Gamma + \mathrm{Id} \otimes \mathbf{c}\}, \tag{4.5}$$

with the natural action of $\mathrm{G}(n\delta)$. The space $\mathrm{Hom}_\Gamma(L \otimes R^n, R^n) \times \mathcal{L}_m \otimes e_\Gamma$ is Poisson by Example 2.15, and there is a moment map for the $\mathrm{G}(n\delta)$-action given by the sum of moment maps for each component, Lemma 2.19. Thus $Y // \mathrm{G}(n\delta)$ is a Marsden-Weinstein reduction.

Lemma 4.11. *There is a Poisson isomorphism*

$$Cal_c // \mathrm{G}(n\delta) = Y // \mathrm{G}(n\delta).$$

Proof. The map $\mathrm{Hom}_\Gamma(L \otimes R^n, R^n) \times \mathcal{L}_m \otimes e_\Gamma \to \mathrm{Hom}_\Gamma(L \otimes R^n, R^n)$ given by projection onto the first component is a $\mathrm{G}(n\delta)$-equivariant Poisson morphism. Now $(\nabla, P) \in Y$ if and only if $[\nabla, \nabla] \in \frac{1}{2} c_1 |\Gamma| \mathcal{O} \otimes e_\Gamma + \mathrm{Id} \otimes \mathbf{c}$ so the projection map induces isomorphisms $Cal_c \cong Y$ and $Cal_c // \mathrm{G}(n\delta) \cong Y // \mathrm{G}(n\delta)$. This latter isomorphism is Poisson by Proposition 2.20. $\square$

Remark 4.12. We note a feature of the definition of Y in (4.5). The variety Y is the fibre of a moment map above the element $-\frac{1}{2} c_1 |\Gamma| \mathrm{Id} \otimes e_\Gamma + \mathrm{Id} \otimes \mathbf{c} \in \mathrm{End}(n\delta) = \mathrm{Lie}\, \hat{\mathrm{G}}(n\delta)^*$ but this element does not lie in $\mathrm{End}(n\delta)_0$. The subgroup $\mathbb{C}^\times \subseteq \hat{\mathrm{G}}(n\delta)$ acts trivially on $\mathrm{Hom}_\Gamma(L \otimes R^n, R^n) \times \mathcal{L}_m \otimes e_\Gamma$ so one might expect that a corresponding moment map will have image in $\mathrm{Lie}(n\delta)_0 = \mathrm{Lie}(\hat{\mathrm{G}}(n\delta)/\mathbb{C}^\times)^* \subset \mathrm{End}(n\delta)$. This does not have to be the case though, see Remark 2.16.

The varieties, $\mathcal{N}(\lambda, \alpha)$, are reductions defined over a single fixed point. The Calogero-Moser space is a Marsden-Weinstein reduction but is defined over coadjoint orbit which a larger than just a single point. Thus to establish Theorem 4.10 we simplify to this latter case. Let

$$\begin{aligned} \hat{Cal}_c = \{(\nabla, I, J) \in \mathrm{Hom}_\Gamma(L \otimes R^n, R^n) \oplus (R^n)^\Gamma \oplus ((R^n)^*)^\Gamma : [\nabla, \nabla] + (I \otimes J) \\ = -\frac{1}{2} c_1 |\Gamma| \operatorname{Id} \otimes e_\Gamma + \operatorname{Id} \otimes \mathbf{c}\}. \end{aligned} \tag{4.6}$$

The Γ-invariant elements of $R^n = \bigoplus_{i=0}^k S_i^{\oplus n\delta_i}$ are the components corresponding to the trivial representation, that is, $(R^n)^\Gamma = S_0^{\oplus n} \cong \mathbb{C}^n$. An action of $\hat{\mathrm{G}}(n\delta)$ on $(R^n)^\Gamma$ is obtained by $(g_i) \cdot v = g_0 v$ for all $(g_i) \in \hat{\mathrm{G}}(n\delta)$ and $v \in \mathbb{C}^n$. Of course, the 0th component of $\hat{\mathrm{G}}(n\delta)$ is simply $\mathrm{GL}(n, \mathbb{C})$ and $(R^n)^\Gamma$ is its natural representation. The vector space $(R^n)^\Gamma \oplus ((R^n)^*)^\Gamma$ is symplectic with form ω, where $\omega((I, J), (I', J')) = -J(I') + J'(I)$. Note that this form differs by a sign from the one defined in Section 1.4 for symplectic vector spaces formed by doubling. The action of $\mathrm{GL}(n, \mathbb{C})$ preserves this form. The corresponding moment map (Theorem 2.18), $(R^n)^\Gamma \oplus ((R^n)^*)^\Gamma \to \mathfrak{gl}_n^*$, is given by $(I, J) \mapsto (A \mapsto \frac{1}{2}\omega(A \cdot (I, J), (I, J)) = J(AI))$. If we compose this with the trace pairing then we get $\nu : (R^n)^\Gamma \oplus ((R^n)^*)^\Gamma \to \mathfrak{gl}_n; \nu(I, J) = I \otimes J$. By Proposition 2.17 ν is $\mathrm{GL}(n, \mathbb{C})$ equivariant and Poisson. Note that ν is a moment map for the Hamiltonian action of $\mathrm{GL}(n, \mathbb{C})$; in Example 2.22 we calculated a moment map for the action of $\mathbb{C}^\times$.

The varieties $\mathrm{Hom}_\Gamma(L \otimes R^n, R^n)$ and $(R^n)^\Gamma \oplus ((R^n)^*)^\Gamma$ are symplectic and both are endowed with a Hamiltonian action by $\hat{\mathrm{G}}(n\delta)$. By Lemma 2.19, $\mathrm{Hom}_\Gamma(L \otimes R^n, R^n) \oplus (R^n)^\Gamma \oplus ((R^n)^*)^\Gamma$ is a symplectic variety and the action of $\hat{\mathrm{G}}(n\delta)$ is Hamiltonian. Therefore the space $\hat{Cal}_c // \hat{\mathrm{G}}(n\delta)$ is a Marsden-Weinstein reduction. We denote the moment map by ρ, explicitly this is

$$\rho : \mathrm{Hom}_\Gamma(L \otimes R^n, R^n) \oplus (R^n)^\Gamma \oplus ((R^n)^*)^\Gamma \to \mathrm{End}(n\delta); (\nabla, I, J) \mapsto [\nabla, \nabla] + I \otimes J.$$

The map ρ is Poisson and $\hat{\mathrm{G}}(n\delta)$-equivariant by Proposition 2.17.

Proposition 4.13. *There is an isomorphism of Poisson varieties* $\hat{Cal}_c // \hat{\mathrm{G}}(n\delta) \cong Cal_c // \mathrm{G}(n\delta)$.

Proof. There is the natural projection of groups $\hat{\mathrm{G}}(n\delta) \to \mathrm{G}(n\delta)$ which induces an inclusion $\mathrm{End}(n\delta)_0 \to \mathrm{End}(n\delta)$. Recall that $m = -\frac{1}{2} n c_1 |\Gamma|$.

Suppose first that $c_1 \neq 0$. Taking traces of the defining equation of $\hat{Cal}_c$,

$$[\nabla, \nabla] + I \otimes J = -\frac{1}{2}c_1|\Gamma| \operatorname{Id} \otimes e_\Gamma + \operatorname{Id} \otimes \mathbf{c}, \tag{4.7}$$

yields $\operatorname{tr}_{R^n}(I \otimes J) = \operatorname{tr}_{R^n}(-\frac{1}{2}c_1|\Gamma| \operatorname{Id} \otimes e_\Gamma) + \operatorname{tr}_{R^n}(\operatorname{Id} \otimes \mathbf{c}) = m$, since $\mathbf{c}$ is traceless on the regular representation. Therefore $I \otimes J \in \mathfrak{gl}_n$ is a rank one matrix with trace m (which implies that it is semisimple).

Recall the notation and results from Examples 2.8 and 2.22. Let $V = \mathbb{C}^n \oplus (\mathbb{C}^n)^* = (R^n)^\Gamma \oplus ((R^n)^*)^\Gamma$. As explained in Example 2.8, the quotient U_m parametrises the $\mathbb{C}^\times$-orbits of the zero set in V of the function $c_1 r_1 + \cdots + c_n r_n - m$. We shall call this zero set V_m. By the previous paragraph $\rho^{-1}(-\frac{1}{2}c_1|\Gamma| \operatorname{Id} \otimes e_\Gamma + \operatorname{Id} \otimes \mathbf{c}) \subseteq \operatorname{Hom}_\Gamma(L \otimes R^n, R^n) \times V_m$. The map ρ is constant on $\mathbb{C}^\times$-orbits so induces a map $\overline{\rho} : \operatorname{Hom}_\Gamma(L \otimes R^n, R^n) \times U_m \to \operatorname{End}(n\delta)$ such that

$$\begin{tikzcd} \operatorname{Hom}_\Gamma(L \otimes R^n, R^n) \times V_m \arrow[d, "\pi_{\mathbb{C}^\times}"'] \arrow[dr, "\rho"] & \\ \operatorname{Hom}_\Gamma(L \otimes R^n, R^n) \times U_m \arrow[r, "\overline{\rho}"'] & \operatorname{End}(n\delta) \end{tikzcd} \tag{4.8}$$

commutes. Here $\pi_{\mathbb{C}^\times}$ is the orbit map for the $\mathbb{C}^\times$ action. Now, ρ is $\hat{\mathrm{G}}(n\delta)$-equivariant and so is $\pi_{\mathbb{C}^\times}$. Therefore the fact that $\pi_{\mathbb{C}^\times}$ is surjective and $\overline{\rho} \circ \pi_{\mathbb{C}^\times} = \rho$ imply that $\overline{\rho}$ is $\hat{\mathrm{G}}(n\delta)$-equivariant also.

On the other hand, recall the isomorphism $\overline{t} : U_m \to \mathcal{L}_m$ from Example 2.22; this is a Poisson $\mathrm{GL}(n, \mathbb{C})$-equivariant isomorphism. We claim that the following diagram commutes

$$\begin{tikzcd} \operatorname{Hom}_\Gamma(L \otimes R^n, R^n) \times U_m \arrow[d, "\operatorname{Id}\times\overline{t}"'] \arrow[dr, "\overline{\rho}"] & \\ \operatorname{Hom}_\Gamma(L \otimes R^n, R^n) \times \mathcal{L}_m \arrow[r, "{[-,-]+\iota}"'] & \operatorname{End}(n\delta), \end{tikzcd}$$

where ι is the embedding $\mathcal{L}_m \hookrightarrow \mathcal{L}_m \otimes e_\Gamma \subset \operatorname{End}(n\delta)$ and $[-,-] + \iota$ is the map taking $(\nabla, M) \in \operatorname{Hom}_\Gamma(L \otimes R^n, R^n) \times \mathcal{L}_m$ to $[\nabla, \nabla] + \iota(M)$. It follows from the surjectivity of $\pi_{\mathbb{C}^\times}$ and (4.8) that it suffices to show that $\rho(\nabla, I, J) = ([-,-] + \iota) \circ (\operatorname{Id} \times \overline{t}) \circ \pi_{\mathbb{C}^\times}(\nabla, I, J)$ for all $(\nabla, I, J) \in \operatorname{Hom}_\Gamma(L \otimes R^n, R^n) \times V_m$. However, $(\operatorname{Id} \times \overline{t}) \circ \pi_{\mathbb{C}^\times}(\nabla, I, J) = (\nabla, I \otimes J)$ which implies that $([-,-] + \iota) \circ (\operatorname{Id} \times \overline{t}) \circ \pi_{\mathbb{C}^\times}(\nabla, I, J) = ([-,-] + \iota)(\nabla, I \otimes J) = [\nabla, \nabla] + I \otimes J = \rho(\nabla, I, J)$ as required. Therefore $\overline{\rho}$ is a moment map for the $\hat{\mathrm{G}}(n\delta)$ action. By Lemma 2.21 the map $\operatorname{Id} \times \overline{t}$ induces a Poisson isomorphism $\overline{\rho}^{-1}(-\frac{1}{2}c_1|\Gamma| \operatorname{Id} \otimes e_\Gamma + \operatorname{Id} \otimes \mathbf{c}) /\!/ \hat{\mathrm{G}}(n\delta) \cong Y /\!/ \mathrm{G}(n\delta)$.

Let S be the coordinate ring of $\mathrm{Hom}_\Gamma(L\otimes R^n, R^n)\oplus V$, let J be the defining ideal of $\mathrm{Hom}_\Gamma(L\otimes R^n, R^n)\times V_m$ and let I be the defining ideal of $\rho^{-1}(-\frac{1}{2}c_1|\Gamma|\,\mathrm{Id}\otimes e_\Gamma+\mathrm{Id}\otimes\mathbf{c})$. We have $J\subseteq I\subseteq R$, and both J and I are $\hat{\mathrm{G}}(n\delta)$-stable. The coordinate ring of $\mathrm{Hom}_\Gamma(L\otimes R^n, R^n)\times U_m$ is $T=S^{\mathbb{C}^\times}/J^{\mathbb{C}^\times}$ and this is a Poisson algebra: it is the coordinate ring for a reduction of the $\mathbb{C}^\times$ action, Example 2.22. By (4.8), the defining ideal of $\overline{\rho}^{-1}(-\frac{1}{2}c_1|\Gamma|\,\mathrm{Id}\otimes e_\Gamma+\mathrm{Id}\otimes\mathbf{c})$ is $I^{\mathbb{C}^\times}/J^{\mathbb{C}^\times}$. Since $\overline{\rho}$ is a moment map for the $\hat{\mathrm{G}}(n\delta)$ action, we can take the reduction $\overline{\rho}^{-1}(-\frac{1}{2}c_1|\Gamma|\,\mathrm{Id}\otimes e_\Gamma+\mathrm{Id}\otimes\mathbf{c})//\,\hat{\mathrm{G}}(n\delta)$ whose coordinate ring is

$$(T/(I^{\mathbb{C}^\times}/J^{\mathbb{C}^\times}))^{\hat{\mathrm{G}}(n\delta)}\cong T^{\hat{\mathrm{G}}(n\delta)}/(I^{\mathbb{C}^\times}/J^{\mathbb{C}^\times})^{\hat{\mathrm{G}}(n\delta)}\cong S^{\hat{\mathrm{G}}(n\delta)}/I^{\hat{\mathrm{G}}(n\delta)} \tag{4.9}$$

(here we are using the reductivity of $\hat{\mathrm{G}}(n\delta)$ several times). Thus there is an isomorphism

$$\overline{\rho}^{-1}(-\frac{1}{2}c_1|\Gamma|\,\mathrm{Id}\otimes e_\Gamma+\mathrm{Id}\otimes\mathbf{c})//\,\hat{\mathrm{G}}(n\delta)\cong\rho^{-1}(-\frac{1}{2}c_1|\Gamma|\,\mathrm{Id}\otimes e_\Gamma+\mathrm{Id}\otimes\mathbf{c})//\,\hat{\mathrm{G}}(n\delta). \tag{4.10}$$

To compare Poisson forms we denote the Poisson brackets of S and T by $\{-,-\}_S$ and $\{-,-\}_T$ respectively. Then for all $t,t'\in T^{\hat{\mathrm{G}}(n\delta)}$ we can write $t=s+J^{\hat{\mathrm{G}}(n\delta)}, t'=s'+J^{\hat{\mathrm{G}}(n\delta)}$ for some $s,s'\in S^{\hat{\mathrm{G}}(n\delta)}$. Now $\{t+(I^{\mathbb{C}^\times}/J^{\mathbb{C}^\times})^{\hat{\mathrm{G}}(n\delta)}, t'+(I^{\mathbb{C}^\times}/J^{\mathbb{C}^\times})^{\hat{\mathrm{G}}(n\delta)}\}=\{t,t'\}_T+I^{\hat{\mathrm{G}}(n\delta)}/J^{\hat{\mathrm{G}}(n\delta)}=(\{s,s'\}_S+J^{\hat{\mathrm{G}}(n\delta)})+I^{\hat{\mathrm{G}}(n\delta)}/J^{\hat{\mathrm{G}}(n\delta)}$. Under the isomorphism (4.9) this last element maps to $\{s,s'\}_S+I^{\hat{\mathrm{G}}(n\delta)}$. It follows that isomorphism (4.10) is Poisson.

Now suppose that $c_1=0$. Then

$$\hat{Cal_c}=\{(\nabla,I,J)\in\mathrm{Hom}_\Gamma(L\otimes R^n,R^n)\oplus(R^n)^\Gamma\oplus((R^n)^*)^\Gamma:[\nabla,\nabla]+(I\otimes J)=\mathrm{Id}\otimes c\}.$$

Consider the closed Poisson subvariety of $\hat{Cal_c}$,

$$Z=\{(\nabla,I,J)\in\hat{Cal_c}:I=J=0\}.$$

It is clear that projecting onto the first component induces a Poisson isomorphism $Z//\,\hat{\mathrm{G}}(n\delta)\cong Cal_c/\,\mathrm{G}(n\delta)$. Therefore we have the following diagram

$$Cal_c//\,\mathrm{G}(n\delta)\cong Z//\,\hat{\mathrm{G}}(n\delta)\hookrightarrow\hat{Cal_c}//\,\hat{\mathrm{G}}(n\delta). \tag{4.11}$$

By Proposition 4.24 there is a surjective algebra homomorphism $\phi^*:\mathcal{O}(Cal_c//\,\mathrm{G}(n\delta))\to Z_c$ where the latter algebra is the centre of a symplectic reflection algebra for a wreath product and in particular has Krull dimension $2n$. Therefore $\mathrm{Dim}\,Cal_c//\,\mathrm{G}(n\delta)\geq 2n$. On the other hand Theorem 4.16 and Lemma 4.17, which we shall prove below, imply that $\hat{Cal_c}//\,\hat{\mathrm{G}}(n\delta)$ is irreducible of dimension $2n$. Therefore $Z//\,\hat{\mathrm{G}}(n\delta)=\hat{Cal_c}//\,\hat{\mathrm{G}}(n\delta)$ and the result follows. □

We state some results we shall need to complete the last stage of the proof of Theorem 4.10. Recall the map ζ from (4.4); dually, the form ω_L defines a Γ-map $\omega_L : L \otimes L \to \mathbb{C}$.

Lemma 4.14. *[19, Lemma 3.1] If M and N are $\mathbb{C}\Gamma$-modules, then there is an isomorphism*

$$\mathrm{Hom}_\Gamma(L \otimes M, N) \to \mathrm{Hom}_\Gamma(M, L \otimes N); \phi \mapsto \phi^\flat$$

where $\phi^\flat = (1 \otimes \phi)(\zeta \otimes 1)$.

The following result is a reformulation of [19, Lemma 3.2]. We mimic the proof therein making the necessary minor changes.

Proposition 4.15. *One can choose maps $\Theta_a \in \mathrm{Hom}_\Gamma(L \otimes S_{t(a)}, S_{h(a)})$ for each $a \in \overline{A}$ in such a way that the θ_a give a basis for each of the spaces $\mathrm{Hom}_\Gamma(L \otimes S_i, S_j)$ and for each $a \in A$*

$$\theta_a \theta_{a^*}^\flat = \frac{1}{\delta_{h(a)}} 1_{S_{h(a)}} \text{ and } \theta_{a^*}\theta_a^\flat = -\frac{1}{\delta_{t(a)}} 1_{S_{t(a)}}.$$

Proof. Recall the extended Dynkin diagrams listed in Section 1.4. Suppose that the underlying graph of Q is of type $\tilde{A_n}$, and without loss of generality we give Q the cyclic orientation. Then $\delta = (1, 1, \ldots, 1, 1)$. The group Γ is cyclic with generator g such that $gx = \eta x$ and $gy = \eta^{-1} y$ where $\eta = e^{\frac{2\pi i}{n}}$. The irreducible representations S_i are one dimensional with g acting as multiplication by η^i. Choose a nonzero element $s_i \in S_i$ for each i. Now if $a \in A$ then $a : i \to i+1$ for some i and we set

$$\theta_a(x \otimes s_i) = s_{i+1},\ \theta_a(y \otimes s_i) = 0;\ \theta_{a^*}(x \otimes s_{i+1}) = 0,\ \theta_{a^*}(y \otimes s_{i+1}) = s_i.$$

These maps clearly satisfy the proposition.

Now suppose that the underlying graph of Q is of type $\tilde{D}$ or $\tilde{E}$. We note that Q is a tree. For each $a \in A$ choose nonzero homomorphisms $\theta_a : L \otimes S_{t(a)} \to S_{h(a)}$, $\theta_{a^*} : L \otimes S_{h(a)} \to S_{t(a)}$. Then $\theta_a \theta_{a^*}^\flat$ and $\theta_{a^*}\theta_a^\flat$ are nonzero endomorphisms of $S_{h(a)}$ and $S_{t(a)}$ respectively. By rescaling the θ_a one can find nonzero scalars m_i for each $i \in I$ such that

$$\theta_a \theta_{a^*}^\flat = m_{h(a)} 1_{S_{h(a)}} \quad \text{and} \quad \theta_{a^*}\theta_a^\flat = -m_{t(a)} 1_{S_{t(a)}}. \tag{4.12}$$

This is achieved as follows. First fix the value m_i for some vertex i. Then, if the m_i have been defined for all vertices in a connected piece of Q and if a is an arrow connecting a vertex i inside the piece and a vertex j outside, one of the two equations above fixes θ_a, θ_{a^*} up to scalar multiples, and the other equation fixes m_j.

Thus $\theta^{\flat}_{a^*}\theta_a$ is $m_{h(a)}$ times the projection of $L\otimes S_{t(a)}$ onto $S_{h(a)}$, and $\theta^{\flat}_a\theta_{a^*}$ is $-m_{t(a)}$ the projection of $L \otimes S_{h(a)}$ onto $S_{t(a)}$. Thus for any vertex i,

$$\sum_{a\in A, t(a)=i} \frac{1}{m_{h(a)}}\theta^{\flat}_{a^*}\theta_a - \sum_{a\in A, h(a)=i} \frac{1}{m_{t(a)}}\theta^{\flat}_a\theta_{a^*} = 1_{L\otimes S_i}. \tag{4.13}$$

Now for any $a \in \overline{A}$ a direct calculation shows that $(\omega_L \otimes 1)(1\otimes\theta^{\flat}_{a^*}\theta_a)(\zeta\otimes 1) = -\theta_{a^*}\theta^{\flat}_a : S_{t(a)} \to S_{t(a)}$. Thus applying this to (4.13) and using (4.12) we obtain

$$\sum_{a\in A, t(a)=i} \frac{m_i}{m_{h(a)}} 1_{S_i} + \sum_{a\in A, h(a)=i} \frac{m_i}{m_{t(a)}} 1_{S_i} = (\omega_L\otimes 1)(1\otimes 1_{L\otimes S_i})(\zeta\otimes 1) = 2\ 1_{S_i}.$$

If we set $n_i = \frac{1}{m_i}$ for each i this yields the equations

$$\sum_{a\in A, t(a)=i} n_{h(a)} + \sum_{a\in A, h(a)=i} n_{t(a)} = 2n_i.$$

Recall the symmetric bilinear form on $\mathbb{Z}^I$ from Section 2.3 which we denoted by $(-,-)$. Extend this to a form to $\mathbb{C}^I$. Let $n \in \mathbb{C}^I$ be the vector with components n_i. Ones sees from the equation above that $(n, \epsilon_i) = 0$ for all i. This forces n to be a multiple of δ, see Section 2.3, and so by rescaling all the θ_a with $a \in A$ we can ensure that $n = \delta$. This implies that $m_i = \frac{1}{\delta_i}$ for all i as required. ◻

The idea of our proof of Theorem 4.16 is based on [51, Section 3].

Theorem 4.16. *There is a Poisson isomorphism* $\hat{Cal}_c//\,\hat{\mathrm{G}}(n\delta) \cong \mathcal{N}(\lambda'(c), \epsilon_\infty + n\delta)$.

Proof. We can identify $(R^n)^\Gamma$ and $((R^n)^*)^\Gamma$ with $\mathrm{Mat}(n\delta_0 \times 1, \mathbb{C})$ and $\mathrm{Mat}(1 \times n\delta_0, \mathbb{C})$ respectively. In this way and by using the decomposition of R^n, (2.5), we can describe

$$\mathrm{Hom}_\Gamma(L\otimes R^n, R^n) \oplus (R^n)^\Gamma \oplus ((R^n)^*)^\Gamma$$

as

$$\mathrm{Hom}_\Gamma(\sum_{m=0}^{k} L\otimes S_m \otimes \mathbb{C}^{n\delta_m}, \sum_{m=0}^{k} S_m\otimes\mathbb{C}^{n\delta_m}) \oplus \mathrm{Mat}(n\delta_0\times 1, \mathbb{C}) \oplus \mathrm{Mat}(1\times n\delta_0, \mathbb{C}).$$

Now from the definition of Q as the quiver obtained from the MacKay graph of Γ we can identify this space with

$$\begin{aligned}\bigoplus_{a\in\overline{A}} &\mathrm{Hom}_\Gamma(L\otimes S_{t(a)}, S_{h(a)}) \otimes \mathrm{Mat}(n\delta_{h(a)}\times n\delta_{t(a)}, \mathbb{C}) \\ &\oplus \mathrm{Mat}(n\delta_0\times 1, \mathbb{C}) \oplus \mathrm{Mat}(1\times n\delta_0, \mathbb{C}).\end{aligned} \tag{4.14}$$

For each arrow $a \in \overline{A}$ we choose $\Theta_a \in \mathrm{Hom}_\Gamma(L \otimes S_{t(a)}, S_{h(a)})$ as in Proposition 4.15. There is a $\hat{\mathrm{G}}(n\delta)$-equivariant isomorphism of vector spaces, Ψ, from

$$\bigoplus_{a\in\overline{A}} \mathrm{Mat}(n\delta_{h(a)} \times n\delta_{t(a)}, \mathbb{C}) \oplus \mathrm{Mat}(n\delta_0 \times 1, \mathbb{C}) \oplus \mathrm{Mat}(1 \times n\delta_0, \mathbb{C})$$

to (4.14) given by

$$\Psi(\underline{B}, i, j) = ((\Theta_a \otimes B_a), i, j).$$

We shall identify the groups $\hat{\mathrm{G}}(n\delta)$ and $\mathrm{Aut}_\Gamma(R^n)$ as in Example 2.9 and this yields an isomorphism of Lie algebras

$$\mathrm{End}(n\delta) \cong \mathrm{End}_\Gamma R^n; (M_i) \mapsto (\mathrm{Id}_{S_i} \otimes M_i). \tag{4.15}$$

Composing the inclusion $\mathbb{C}^I \hookrightarrow \mathrm{End}(n\delta); (\lambda_i) \mapsto (\lambda_i \mathrm{Id}_{n\delta_i})$ with (4.15) yields an inclusion $\mathbb{C}^I \hookrightarrow \mathrm{End}_\Gamma R^n$. The image of $\lambda'(c)$ under this map is $\bigoplus_{m\in I} \delta_m(-\frac{1}{2}c_1|\Gamma|\mathrm{Id} \otimes e_\Gamma + \mathrm{Id} \otimes \mathbf{c})|_{S_m \otimes \mathbb{C}^{n\delta_m}} \in \mathrm{End}_\Gamma R^n$.

We claim that Ψ maps $\mu^{-1}_{\epsilon_\infty + n\delta}(\lambda'(c))$ to $\hat{Cal}_c$. Let $(\underline{B}, i, j) \in \mu^{-1}_{\epsilon_\infty+n\delta}(\lambda'(c))$, then

$$\mu_{\epsilon_\infty+n\delta}(\underline{B}, i, j) = \bigoplus_{m=0}^{k} [\sum_{a\in A, h(a)=m} B_a B_{a^*} - \sum_{a\in A, t(a)=m} B_{a^*} B] + ij - ji = (-\lambda'(c)\cdot\delta, \lambda'(c)). \tag{4.16}$$

Here we have $ij \in \mathrm{Mat}(n\delta_0 \times n\delta_0, \mathbb{C})$ and $ji \in \mathrm{Mat}(\epsilon_\infty \times \epsilon_\infty, \mathbb{C})$. We note that $ji = \mathrm{tr}_{\mathbb{C}^n} ij$ so that the term at the ∞ vertex, $-ji = -\lambda'(c)\cdot\delta$, is implied by the terms at the other vertices. Therefore we omit this last term in the following.

Using the isomorphism (4.15) we obtain

$$\bigoplus_{m=0}^{k} [1_{S_m} \otimes (\sum_{a\in A, h(a)=m} B_a B_{a^*} - \sum_{a\in A, t(a)=m} B_{a^*} B_a)] + 1_{S_0} \otimes (i \otimes j)$$

$$= \bigoplus_{m=0}^{k} \delta_m(-\frac{1}{2}c_1|\Gamma|\mathrm{Id} \otimes e_\Gamma + \mathrm{Id} \otimes \mathbf{c})|_{S_m \otimes \mathbb{C}^{n\delta_m}} \in \mathrm{End}_\Gamma R^n,$$

so to prove the claim we need only verify that for each m,

$$[\Psi(\underline{B}), \Psi(\underline{B})]|_{S_m \otimes \mathbb{C}^{n\delta_m}} = \frac{1}{\delta_m} 1_{S_m} \otimes (\sum_{a\in A, h(a)=m} B_a B_{a^*} - \sum_{a\in A, t(a)=m} B_{a^*} B_a).$$

Let $r_m \otimes v_m \in S_m \otimes \mathbb{C}^{n\delta_m}$, then

$$
\begin{aligned}
&[\Psi(\underline{B}), \Psi(\underline{B})](r_m \otimes v_m) = \Psi(\underline{B_a}) \circ (\mathrm{Id} \otimes \Psi(\underline{B}))((x \otimes y - y \otimes x) \otimes r_m \otimes v_m) \\
&= \Psi(\underline{B})\Big(\sum_{\substack{a \in \overline{A}, \\ t(a)=m}} x \otimes \Theta_a(y \otimes r_m) \otimes B_a(v_m) - \sum_{\substack{a \in \overline{A}, \\ t(a)=m}} y \otimes \Theta_a(x \otimes r_m) \otimes B_a(v_m)\Big) \\
&= \sum_{\substack{a \in \overline{A}, \\ t(a)=m}} \sum_{\substack{b \in \overline{A}, \\ t(b)=h(a)}} \Theta_b(x \otimes \Theta_a(y \otimes r_m)) \otimes B_b(B_a(v_m)) \\
&\qquad\qquad - \sum_{\substack{a \in \overline{A}, \\ t(a)=m}} \sum_{\substack{b \in \overline{A}, \\ t(b)=h(a)}} \Theta_b(y \otimes \Theta_a(x \otimes r_m)) \otimes B_b(B_a(v_m)) \\
&= \sum_{\substack{a \in \overline{A}, \\ t(a)=m}} \sum_{\substack{b \in \overline{A}, \\ t(b)=h(a)}} \big[\Theta_b(x \otimes \Theta_a(y \otimes r_m) - y \otimes \Theta_a(x \otimes r_m))\big] \otimes B_b(B_a(v_m)).
\end{aligned}
\tag{4.17}
$$

Because $[\Psi(\underline{B}), \Psi(\underline{B})]$ is a Γ-map we must have that the b's appearing in (4.17) have $h(b) = m$ and since Q is a simply laced quiver it follows that $b = a^*$. Therefore (4.17) equals:

$$
\begin{aligned}
&\sum_{\substack{a \in \overline{A}, \\ t(a)=m}} \big[\Theta_{a^*}(x \otimes \Theta_a(y \otimes r_m) - y \otimes \Theta_a(x \otimes r_m))\big] \otimes B_{a^*}(B_a(v_m)) \\
&= \sum_{\substack{a \in \overline{A}, \\ t(a)=m}} \big[\Theta_{a^*}(1 \otimes \Theta_a)(x \otimes y \otimes r_m - y \otimes x \otimes r_m)\big] \otimes B_{a^*}(B_a(v_m)) \\
&= \sum_{\substack{a \in \overline{A}, \\ t(a)=m}} \Theta_{a^*}\Theta_a^{\flat}(r_m) \otimes B_{a^*}(B_a(v_m)).
\end{aligned}
$$

Now by Proposition 4.15 this equals

$$
\Big(\frac{1}{\delta_m} 1_{S_m} \otimes \Big(\sum_{a \in \overline{A}, h(a)=m} B_a B_{a^*} - \sum_{a \in A, t(a)=m} B_{a^*} B_a\Big)\Big)(r_m \otimes v_m), \tag{4.18}
$$

as required. It is clear from (4.18) that $\Psi(\underline{B}, i, j) \in \hat{Cal}_c$ if and only if $(\underline{B}, i, j) \in \mu^{-1}_{\epsilon_\infty + n\delta}(\lambda'(c))$. Thus the map Ψ induces an isomorphism of $\mu^{-1}_{\epsilon_\infty + n\delta}(\lambda'(c))$ with $\hat{Cal}_c$, and since Ψ is equivariant for the group actions we get an induced isomorphism

$$
\tilde{\Psi} : \mathcal{N}\left(\lambda'(c), \epsilon_\infty + n\delta\right) \to \hat{Cal}_c // \hat{\mathrm{G}}(n\delta).
$$

We now compare Poisson structures. For all $(\underline{B}, i, j), (\underline{B}', i', j') \in \mathrm{Rep}(\overline{Q'}, \epsilon_\infty + n\delta)$,

$$
\omega((\underline{B}, i, j), (\underline{B}', i', j')) = \sum_{i \in I} \Big(\sum_{a \in A; t(a)=i} - \mathrm{tr}\,(B_{a^*} B'_a) + \sum_{a \in A; h(a)=i} \mathrm{tr}\,(B'_{a^*} B_a)\Big) + tr(j'i - ji')
$$

which by an easy application of (4.18) is equal to

$$\sum_{i\in I} tr_{S_i\otimes\mathbb{C}^{n\delta_i}}[\Psi(\underline{B}),\Psi(\underline{B}')] + tr(i\otimes j' - j\otimes i').$$

Therefore by (2.10) and (4.3) Ψ intertwines symplectic forms. Thus $\tilde{\Psi}$ is a Poisson isomorphism by Lemma 2.21. $\square$

Finally we can combine our results.

Proof of Theorem 4.10. The isomorphism is obtained composing those of Proposition 4.13 and Theorem 4.16. Since both of these are Poisson maps their composition is also. $\square$

We state a result concerning $\mathcal{N}(\lambda'(c), \epsilon_\infty + n\delta)$ which is independent of the results of this section.

Lemma 4.17. *The variety $\mathcal{N}(\lambda'(c), \epsilon_\infty + n\delta)$ is irreducible variety of dimension $2n$*

Proof. By Theorem 2.30 (iii), this variety is irreducible and has dimension $2n$ as long as $\epsilon_\infty + n\delta$ is a sum of elements from $R^+_{\lambda'(c)}$ and $|\epsilon_\infty + n\delta|_{\lambda'(c)} = n$. Let $\beta = \epsilon_\infty + n\delta$. One sees that β is in fact a root: $(\epsilon_i, \beta) = 0$ for all $i \notin \{0, \infty\}$, $(\epsilon_0, \beta) = (\epsilon_0, \epsilon_\infty) = -1$ and $(\epsilon_\infty, \beta) = 2 - n\delta_0 \leq 0$. By Lemma 2.31 and Theorem 2.30 (i), $|\beta|_{\lambda'(c)} = p'(\beta)$ and, as noted in Lemma 2.31, $p'(\beta) = n$. $\square$

Now using the theorems from this section we get the following.

Corollary 4.18. *The variety $Cal_c//\,\mathrm{G}(n\delta)$ is irreducible of dimension $2n$.*

Proof. By Theorem 4.10, $Cal_c//\,\mathrm{G}(n\delta) \cong \mathcal{N}(\lambda'(c), \epsilon_\infty + n\delta)$. Thus the result follows from Lemma 4.17. $\square$

4.3 An isomorphism between X_c and $Cal_c//\,\mathrm{G}(n\delta)$

Having established that $Cal_c//\,\mathrm{G}(n\delta) \cong \mathcal{N}(\lambda'(c), \epsilon_\infty + n\delta)$, we show that X_c, the variety associated to H_c for a wreath product, is isomorphic to $Cal_c//\,\mathrm{G}(n\delta)$. Combining these results we obtain an isomorphism between a Marsden-Weinstein reduction for a quiver and the variety X_c, Corollary 4.25. Furthermore, we can identify symplectic leaves of each variety.

The contents of this section are based closely on those of [26, Section 11]. We modify the arguments of Etingof and Ginzburg slightly and to ensure that the arguments used there pass to our situation we provide full details. We shall state clearly where a result is not the author's.

Let $Cal_c//\mathrm{G}(n\delta)$ be the Calogero-Moser space for Γ_n and X_c the variety associated to a symplectic reflection algebra for Γ_n. We use the notation and hypotheses of Example 2.9 and Notation 1.22. In particular L is a two dimensional symplectic vector space with symplectic basis $\{x, y\}$. Let $n > 1$. For a finite group $\Gamma \subset SL(L)$ let Γ_n denote its wreath product with S_n which acts on $V = L^{\oplus n}$ preserving the symplectic form $\omega = \omega_L^{\oplus n}$. Let e, e' be the symmetrising idempotents of $\mathbb{C}\Gamma, \mathbb{C}\Gamma_n$ respectively. Let $S_{n-1} \ltimes \Gamma^{n-1} = \Gamma_{n-1} < \Gamma_n$, where S_{n-1} fixes the label $1 \in \{1, \ldots, n\}$. Let $f \in \mathbb{C}\Gamma_{n-1}$ be the symmetrising idempotent. We let Γ_{n-1} act on Γ_n by multiplication on the left.

Lemma 4.19. *[26, (11.14)] The subspace, $\mathbb{C}\Gamma_n^{\Gamma_{n-1}}$, of Γ_{n-1}-fixed points in $\mathbb{C}\Gamma_n$ is isomorphic to $\mathbb{C}^n \otimes \mathbb{C}\Gamma$ as a Γ-module.*

Proof. The action of $\gamma \in \Gamma$ on $\mathbb{C}\Gamma_n^{\Gamma_{n-1}}$ is given by multiplication on the left by $\gamma_1 \in \mathbb{C}\Gamma_n$. A basis of $\mathbb{C}\Gamma_n^{\Gamma_{n-1}}$ is given by $\{f \cdot \gamma_1 \cdot s_{1j} : \gamma \in \Gamma, 1 \leq j \leq n\}$. If $v_1, \ldots, v_n$ is the standard basis of $\mathbb{C}^n$ then the isomorphism $\mathbb{C}^n \otimes \mathbb{C}\Gamma \to \mathbb{C}\Gamma_n^{\Gamma_{n-1}}$ is given by

$$v_j \otimes \gamma \mapsto f \cdot \gamma_1 \cdot s_{1j}. \tag{4.19}$$

□

The regular representation of Γ_n plays an important role in our description of X_c. Let $\mathrm{Rep}_{\mathbb{C}\Gamma_n} H_c$ denote the set of all algebra homomorphisms $H_c \to \mathrm{End}_{\mathbb{C}} \mathbb{C}\Gamma_n$ such that the restriction to $\mathbb{C}\Gamma_n \subset H_c$ is the regular representation of Γ_n. This has the structure of an affine algebraic variety and each point corresponds to a representation of H_c which is isomorphic, as a $\mathbb{C}\Gamma_n$-module, to the regular representation. Thus we can think of a point $M \in \mathrm{Rep}_{\mathbb{C}\Gamma_n} H_c$ as an H_c-module, and we can consider the one dimensional vector subspace $e'M$ of M (where we view $e' \in \mathbb{C}\Gamma_n \subset H_c$). Therefore we have a morphism of algebraic varieties,

$$\pi : \mathrm{Rep}_{\mathbb{C}\Gamma_n} H_c \to X_c; \ M \mapsto \mathrm{ann}_{Z_c} e'M,$$

which is well defined because Z_c commutes with e'. The reductive algebraic group $\mathrm{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ acts naturally on $\mathrm{Rep}_{\Gamma_n}(H_c)$ and this action preserves the fibres of π.

Proposition 4.20. *[26, Theorem 3.7 (i)] There is a unique irreducible component of* $\mathrm{Rep}^o_{\mathbb{C}\Gamma_n} H_c$ *of* $\mathrm{Rep}_{\mathbb{C}\Gamma_n} H_c$ *whose image under* π *is dense in* X_c. *Furthermore, there is an isomorphism* $\pi^\sharp$: $Z_c \to \mathcal{O}(\mathrm{Rep}^o_{\mathbb{C}\Gamma_n} H_c)^{\mathrm{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)}$.

Let E be an element of $\mathrm{Rep}_{\mathbb{C}\Gamma_n} H_c$. For $v \in L$, the element $v_1 = (v, 0, \dots, 0) \in L^{\oplus n} \subset H_c$ commutes with $\Gamma_{n-1} \subset \Gamma_n$. Therefore the elements x_1 and y_1 preserve the subspace $E^{\Gamma_{n-1}}$, and the map $L \to \mathrm{End}_{\mathbb{C}}(E^{\Gamma_{n-1}}); v \mapsto v_1|_{E^{\Gamma_{n-1}}}$ is Γ-equivariant. The following lemma is identical to [26, Lemma 11.15] except for the factor of $\frac{1}{2}$ which appears below. We provide a proof to justify this difference. Let $P \in \mathrm{Mat}(n, \mathbb{C})$ be the matrix

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix}. \tag{4.20}$$

Lemma 4.21. *The endomorphism* $[x_1, y_1]|_{E^{\Gamma_{n-1}}}$ *corresponds under the isomorphism of (4.19) to the endomorphism of* $\mathbb{C}^n \otimes \mathbb{C}\Gamma$ *given by* $\frac{1}{2}c_1|\Gamma|P \otimes e_\Gamma + Id_{\mathbb{C}^n} \otimes \mathbf{c}$.

Proof. Recall the conjugacy classes of symplectic reflections of Γ_n listed in Section 1.4. For any $\gamma \in \Gamma$ and any i, j we consider the action of $1 - s_{ij} \cdot \gamma_i \cdot \gamma_j^{-1}$ on V. By applying this map to the basis $\{x_k, y_k : k \neq j\} \cup \{x_j^\gamma, y_j^\gamma\}$ one sees that Ker $(1 - s_{ij} \cdot \gamma_i \cdot \gamma_j^{-1})$ is spanned by $\{x_k, y_k : k \neq i, j\} \cup \{x_i + x_j^\gamma, y_i + y_j^\gamma\}$ and Im $(1 - s_{ij} \cdot \gamma_i \cdot \gamma_j^{-1})$ is spanned by $\{x_i - x_j^\gamma, y_i - y_j^\gamma\}$. It follows that $\omega_{s_{ij} \cdot \gamma_i \cdot \gamma_j^{-1}}(x_1, y_1) = 0$ if $i, j \neq 1$. Writing $x_1 = \frac{1}{2}(x_1 + x_j^\gamma) + \frac{1}{2}(x_1 - x_j^\gamma)$ and $y_1 = \frac{1}{2}(y_1 + y_j^\gamma) + \frac{1}{2}(y_1 - y_j^\gamma)$ we see that

$$\omega_{s_{1j} \cdot \gamma_1 \cdot \gamma_j^{-1}}(x_1, y_1) = \omega(\frac{1}{2}(x_1 - x_j^\gamma), \frac{1}{2}(y_1 - y_j^\gamma)) = \frac{1}{4}(\omega(x_1, y_1) + \omega(x_j^\gamma, y_j^\gamma)) = \frac{1}{2}.$$

Also, it is easy to see that for $\gamma \in \Gamma \setminus \{1\}$

$$\omega_{\gamma_i}(x_1, y_1) = \begin{cases} 0 \text{ if } i \neq 1 \\ \omega_L(x, y) \text{ if } i = 1. \end{cases}$$

Therefore the relations for H_c in (1.2) imply that

$$\begin{aligned} [x_1, y_1] &= c_1 \sum_{\gamma \in \Gamma, j > 1} \omega_{s_{1j} \cdot \gamma_1 \cdot \gamma_j^{-1}}(x_1, y_1) s_{1j} \cdot \gamma_1 \cdot \gamma_j^{-1} + \sum_{\gamma \in \Gamma \setminus \{1\}} \underline{c}(\gamma) \omega_{\gamma_1}(x_1, y_1) \gamma_1 \\ &= \frac{1}{2} c_1 \sum_{\gamma \in \Gamma, j > 1} s_{1j} \cdot \gamma_1 \cdot \gamma_j^{-1} + \sum_{\gamma \in \Gamma \setminus \{1\}} \underline{c}(\gamma) \gamma_1. \end{aligned} \tag{4.21}$$

The element $\sum_{\gamma\in\Gamma\setminus\{1\}} \underline{c}(\gamma)\gamma_1$ commutes with f so that in the basis of (4.19), $\sum_{\gamma\in\Gamma\setminus\{1\}} \underline{c}(\gamma)\gamma_1|_{E^{\Gamma_{n-1}}} = \mathrm{Id}_{\mathbb{C}^n} \otimes \mathbf{c}$. Therefore by Lemma 4.22 below we have

$$[x_1, y_1]|_{E^{\Gamma_{n-1}}} = \frac{1}{2}c_1|\Gamma|P \otimes e_\Gamma + \mathrm{Id}_{\mathbb{C}^n} \otimes \mathbf{c}$$

as required. □

We require a calculation for the proof of Lemma 4.21. Let $e_i = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \gamma_i \in \mathbb{C}\Gamma_n$ and let $\hat{e_i} = \prod_{j\neq i} e_j$. Let $S = \frac{1}{(n-1)!}(\sum_{\substack{\tau\in S_n \\ \tau(1)=1}} \tau)$. Thus f equals $S \cdot \hat{e_1}$.

Lemma 4.22. *The endomorphism $\sum_{\gamma\in\Gamma, j>1} s_{1j} \cdot \gamma_1 \cdot \gamma_j^{-1}|_{E^{\Gamma_{n-1}}}$ corresponds, in the basis of (4.19), to the endomorphism $|\Gamma|P \otimes e_\Gamma$ of $\mathbb{C}^n \otimes \mathbb{C}\Gamma$.*

Proof. We note a few easy facts which we shall use. For all $\gamma \in \Gamma$ and $i, j > 1$ we have $\gamma_i \cdot f = f$ and $s_{ij} \cdot S = S$. For any i the product $\hat{e_i} \cdot e_i = e_i \cdot \hat{e_i} = \prod_{1\leq j\leq n} e_j$ is central in $\mathbb{C}\Gamma_n$. Also, S commutes with both $\sum_{j>1} s_{1j}$ and e_1.

We calculate the product of $\sum_{\gamma\in\Gamma, j>1} s_{1j} \cdot \gamma_1 \cdot \gamma_j^{-1}$ with a typical basis element $f \cdot g_1 \cdot s_{1i}$:

$$\begin{aligned}
(\sum_{\gamma\in\Gamma, j>1} s_{1j} \cdot \gamma_1 \cdot \gamma_j^{-1})(f \cdot g_1 \cdot s_{1i}) &= \sum_{\gamma\in\Gamma, j>1} s_{1j} \cdot \gamma_1 \cdot f \cdot g_1 \cdot s_{1i} = \sum_{\gamma\in\Gamma, j>1} s_{1j} \cdot \gamma_1 \cdot S \cdot \hat{e_1} \cdot g_1 \cdot s_{1i} \\
= \sum_{j>1} s_{1j} \cdot (\sum_{\gamma\in\Gamma} \gamma_1) \cdot S \cdot \hat{e_1} \cdot g_1 \cdot s_{1i} &= |\Gamma| \sum_{j>1} s_{1j} \cdot e_1 \cdot S \cdot \hat{e_1} \cdot g_1 \cdot s_{1i} = |\Gamma| \sum_{j>1} s_{1j} \cdot S \cdot \hat{e_1} \cdot e_1 \cdot s_{1i} \\
&= |\Gamma| \sum_{j>1} S \cdot \hat{e_1} \cdot e_1 \cdot s_{1j} \cdot s_{1i}.
\end{aligned}$$

If $i = 1$ then the last line is equal to $|\Gamma| \sum_{j\neq 1} f \cdot e_1 \cdot s_{1j}$. If $i \neq 1$ then the last line equals

$$\begin{aligned}
|\Gamma| S \cdot \hat{e_1} \cdot e_1 + |\Gamma| \sum_{\substack{j>1 \\ j\neq i}} S \cdot \hat{e_1} \cdot e_1 \cdot s_{ij} \cdot s_{1j} &= |\Gamma| f \cdot e_1 + |\Gamma| \sum_{\substack{j>1 \\ j\neq i}} S \cdot s_{ij} \cdot \hat{e_1} \cdot e_1 \cdot s_{1j} \\
&= |\Gamma| f \cdot e_1 + |\Gamma| \sum_{\substack{j>1 \\ j\neq i}} f \cdot e_1 \cdot s_{1j} = |\Gamma| \sum_{j\neq i} f \cdot e_1 \cdot s_{1j}.
\end{aligned}$$

□

The theorem we now prove below is based very closely on [26, Theorem 11.16].

Theorem 4.23. *X_c and $Cal_c // \mathrm{G}(n\delta)$ are isomorphic as Poisson varieties (up to nonzero scalar multiple).*

Proof. We first show that $Cal_c//\,\mathrm{G}(n\delta)$ is isomorphic to X_c. Our argument follows that of [26, Theorem 11.16] which proves that these varieties are isomorphic for generic values of c. We show that the generic hypothesis can be removed.

By [26, pp.311], there is a morphism

$$\psi : \mathrm{Rep}_{\Gamma_n}(H_c) \longrightarrow Cal_c.$$

This morphism is obtained as follows. In the notation of the paragraph preceeding Lemma 4.21, for any module $M \in \mathrm{Rep}_{\Gamma_n}(H_c)$ we have a Γ-equivariant map

$$\nabla_M : L \to \mathrm{End}_{\mathbb{C}}(M^{\Gamma_{n-1}});\ v \mapsto v_1|_{M^{\Gamma_{n-1}}}.$$

We set $\psi(M) = \nabla_M$, by Lemma 4.21 this is a well-defined map.

Any $\rho \in \mathrm{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ restricts to an automorphism $(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}} \to (\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}$. There is an action of Γ on $(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}$ defined for all $\gamma \in \Gamma$ by multiplication by γ_1. The restriction of ρ to Γ_{n-1}-invariants is equivariant for this action and we therefore obtain a homomorphism of groups $\mathrm{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n) \to \mathrm{Aut}_{\Gamma}(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}} \twoheadrightarrow \mathrm{G}(n\delta)$.

The morphism ψ intertwines the $\mathrm{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ and $\mathrm{G}(n\delta)$ actions, and hence induces a map

$$\phi^\sharp : \mathcal{O}(Cal_c//\,\mathrm{G}(n\delta)) \longrightarrow \mathcal{O}(\mathrm{Rep}^o_{\Gamma_n}(H_c)^{\mathrm{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)} \cong Z_c. \tag{4.22}$$

The argument of [26, Theorem 11.16] applies verbatim to show that $\phi^\sharp$ is surjective. For the sake of completeness we write out the details in Proposition 4.24 below.

The proof given in [26, Theorem 11.16] that $\phi^\sharp$ is injective requires the generic hypothesis on c. We circumvent this in the following way: by Corollary 4.18 $Cal_c//\,\mathrm{G}(n\delta)$ is an irreducible variety of dimension $2n$. The variety X_c is also irreducible of dimension $2n$ and therefore the surjective morphism $\phi^\sharp$ must be injective also. Therefore $Cal_c//\,\mathrm{G}(n\delta) \cong X_c$.

To see that this isomorphism is Poisson (up to nonzero scalar multiple) we use the argument of the penultimate two paragraphs of [26, Theorem 11.16]. We first take filtrations of $\mathcal{O}(Cal_c//\,\mathrm{G}(n\delta))$ and Z_c. The former algebra is the quotient by an ideal of the ring of invariants of polynomial functions on a vector space, and so has a canonical filtration which we call $\mathcal{F}$. On the other hand we have already seen that Z_c inherits a filtration (which we called $\mathcal{Z}$ in the paragraph before Proposition 1.14) from the one on H_c. By the proof Proposition 4.24 the map $(\phi^\sharp)^{-1} : Z_c \to \mathcal{O}(Cal_c//\,\mathrm{G}(n\delta))$

takes algebra generators of Z_c to elements of $\mathcal{O}(Cal_c// \mathrm{G}(n\delta))$ of the same degree. Therefore for all i, $(\phi^\sharp)^{-1}(\mathcal{Z}_i) \subseteq \mathcal{F}_i$.

We have seen in the proof of Theorem 4.16 that there is a $\mathrm{G}(n\delta)$-equivariant isomorphism between $\mathrm{Hom}_\Gamma(L \otimes R^n, R^n)$ and $\mathrm{Rep}(Q, n\delta)$. It is a result of LeBruyn and Procesi that the coordinate ring of $\mathrm{Rep}(Q,n\delta)// \mathrm{G}(n\delta)$ is generated by traces of oriented cycles, [53, Theorem 2]. Using this, and because one can view $\mathcal{O}(Cal_c// \mathrm{G}(n\delta))$ as a quotient of the coordinate ring of $\mathrm{Hom}_\Gamma(L \otimes R^n, R^n)// \mathrm{G}(n\delta)$, we see that the coordinate ring of $Cal_c// \mathrm{G}(n\delta)$ is generated by traces of monomials. This means that $\mathcal{O}(Cal_c// \mathrm{G}(n\delta))$ is generated by functions of the form $\nabla \mapsto \mathrm{tr}_{R^n} F(\nabla(x), \nabla(y))$ where F is a noncommutative monomial in two variables. Let g be such a function and suppose that g has filtration degree d. Let E be an irreducible H_c-module isomorphic to the regular representation of Γ_n (see Theorem 1.17). Then $\phi^\sharp(g)|_E$ is scalar multiplication by $\mathrm{tr}_E F(x_1, y_1) \cdot f$ where $F(x_1, y_1) \in H_c$ has filtration degree d. Now $\phi^\sharp(g)$ has filtration degree $\leq d$ by [26, Lemma 11.18] and so $\phi^\sharp(\mathcal{F}_i) \subseteq \mathcal{Z}_i$ for all i. We conclude that $\phi^\sharp(\mathcal{F}_i) = \mathcal{Z}_i$ for all i.

Let $\{-,-\}, \{-,-\}'$ be the Poisson brackets on Z_c and $\mathcal{O}(Cal_c// \mathrm{G}(n\delta))$ respectively. Since $\{-,-\}'$ is induced from a symplectic form on a vector space the degree of $\{-,-\}'$ is less than or equal to -2. For some $\lambda \in \mathbb{C}$, $\phi^\sharp\{-,-\}' = \lambda\{-,-\}$ by [26, Lemma 2.26]. In fact $\lambda \neq 0$ because $\phi^\sharp$ is an isomorphism and $\{-,-\}'$ is not the zero bracket. $\square$

One consequence of the above theorem is that the isomorphism $X_c \cong Cal_c// \mathrm{G}(n\delta)$ maps symplectic leaves to symplectic leaves by Lemma 3.6.

We shall need an alternative description of $Cal_c// \mathrm{G}(n\delta)$ in order to prove Proposition 4.24. Recall the matrix P from (4.20). Let $G(P) \subset \mathrm{G}(n\delta)$ be the stabiliser of $P \in \mathrm{End}(n\delta)$. Define

$$Cal_{P,c} = \{\nabla \in \mathrm{Hom}_\Gamma(L, \mathrm{End}_{\mathbb{C}}(R^n)) : [\nabla(x), \nabla(y)] \in \frac{1}{2}c_1|\Gamma| P \otimes e_\Gamma + \mathrm{Id} \otimes \mathbf{c}\}.$$

By Lemma 2.5 $G(P)$ is a reductive algebraic group and, as noted in the line preceding [26, (11.6)], $Cal_c// \mathrm{G}(n\delta) \cong Cal_{P,c}//G(P)$.

We can embed $G(P)$ into $\hat{\mathrm{G}}(n\delta)$ as follows. Define the group $\hat{\mathrm{G}}'(n\delta) = \prod_{i \neq 0} \mathrm{GL}(n\delta_i, \mathbb{C})$. Thus $\mathrm{G}(n\delta) = \mathrm{GL}(n, \mathbb{C}) \times \hat{\mathrm{G}}'(n\delta)/\mathbb{C}^\times$. The decomposition $\mathbb{C}^n = \mathrm{Ker}(P + \mathrm{Id}) \oplus \mathrm{Im}(P + \mathrm{Id})$ gives an embedding $\iota : \mathrm{GL}(n-1, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$; $A \mapsto A \oplus \mathrm{Id}_{\mathrm{Im}(P+\mathrm{Id})}$. The map

$$\mathrm{GL}(n-1, \mathbb{C}) \times \hat{\mathrm{G}}'(n\delta) \xrightarrow{\iota \times \mathrm{Id}} \mathrm{GL}(n, \mathbb{C}) \times \hat{\mathrm{G}}'(n\delta) \longrightarrow \hat{\mathrm{G}}(n\delta)/\mathbb{C}^\times$$

induces an isomorphism of the group $\mathrm{GL}(n-1,\mathbb{C})\times\hat{\mathrm{G}}'(n\delta)$ with $G(P)$. We invert this isomorphism to obtain an embedding of $G(P)$ in $\hat{\mathrm{G}}(n\delta)$.

Proposition 4.24. *[26, Theorem 11.16] The algebra homomorphism $\phi^\sharp$ from (4.22) is surjective.*

Proof. Let SL and TL denote the symmetric and tensor algebras of L respectively. Let $sym : SL \to TL$ denote the symmetrisation map. For all m and all $a_1,\dots,a_m \in L$, sym maps a monomial $a_1\dots a_m \in SL$ to $\frac{1}{m!}\sum_{\sigma\in S_m} a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(m)} \in TL$. Define a map $f : TL \to TL^{\otimes n}$ by $f(a)=\sum_{i=1}^n 1^{\otimes(i-1)}\otimes a\otimes 1^{\otimes(n-i)}$. We obtain a map $SL\to H_c$ given by the following composition:

$$SL \xrightarrow{sym} TL \xrightarrow{f} TL^{\otimes n} = T(L^{\oplus n}) \longrightarrow H_c\,, \tag{4.23}$$

where the final arrow is the projection map. For any $p\in SL$ denote its image in (4.23) by a_p. As usual let e' be the symmetrising idempotent in $\mathbb{C}\Gamma_n$. We define a map $\sigma : SL^\Gamma \to e'H_ce'$ by $f\mapsto e'a_fe'$ (which is not, in general, an algebra homomorphism). We obtain a filtration on SL^Γ via its natural grading, Theorem 1.1 (i), with the result that its associated graded algebra is equal to SL^Γ. The algebra $e'H_ce'$ has a filtration, $\mathcal{E}$, defined in Section 1.3 so that $\mathrm{gr}_{\mathcal{E}}e'H_ce'\cong SV^{\Gamma_n} = ((SL^\Gamma)^{\otimes n})^{S_n}$ by Corollary 1.11. It follows from the definition of $\mathcal{E}$ that σ is a filtration preserving map. Let $\mathrm{gr}\sigma : SL^\Gamma \to ((SL^\Gamma)^{\otimes n})^{S_n}$ be the associated graded map. Corollary 1.11 implies that for all $p\in SL^\Gamma$, $\mathrm{gr}\sigma(p)=\overline{f(sym(p))}$ where the right hand term denotes the image of $f(sym(p))$ in $((SL^\Gamma)^{\otimes n})^{S_n}$ under the natural quotient map. Therefore the image of $\mathrm{gr}\sigma$ generates $((SL^\Gamma)^{\otimes n})^{S_n}$ by [26, Lemma 11.17] and so the image of σ generates $e'H_ce'$ by Corollary 1.11.

Using the basis $\{x,y\}$ of L we identify SL with the polynomial algebra $\mathbb{C}[x,y]$. Thus, composing σ with the inverse of the Satake isomorphism gives a map $\sigma^\sharp : \mathbb{C}[x,y]^\Gamma = SL^\Gamma \to Z_c$. By the previous paragraph the image of $\sigma^\sharp$ generates Z_c.

Let $\mathbb{C}_{\mathrm{diag}}\in\mathbb{C}^n$ denote the principal diagonal, that is, the subspace of fixed points under the action of S_n by permutation. Let $e_{\mathrm{diag}} : \mathbb{C}^n \to \mathbb{C}_{\mathrm{diag}} \subset \mathbb{C}^n$ be the map given by the matrix $P+\mathrm{Id}$. Then $e_{\mathrm{diag}}\otimes e_\Gamma$ is the projection of $R^n=\mathbb{C}^n\otimes\mathbb{C}\Gamma$ onto the one dimensional subspace $\mathbb{C}_{\mathrm{diag}}\otimes\mathbb{C}e$. For all $f\in\mathbb{C}[x,y]^\Gamma$ and $\nabla\in Cal_{P,c}$ consider the function

$$F : \nabla \mapsto (e_{\mathrm{diag}}\otimes e_\Gamma \circ \mathrm{sym}(f)(\nabla(x),\nabla(y)) \circ e_{\mathrm{diag}}\otimes e_\Gamma) \in \mathbb{C}e_{\mathrm{diag}}\otimes e_\Gamma = \mathbb{C}. \tag{4.24}$$

We note that $\mathbb{C}_{\mathrm{diag}}\otimes\mathbb{C}e$ is fixed by $G(P)\subset\hat{\mathrm{G}}(n\delta)$ and so F is a $G(P)$-invariant function. To calculate $\phi^\sharp$ of this function at a point $m\in X_c=\operatorname{Max} Z_c$ we first take an element of $M\in\mathrm{Rep}^o_{\mathbb{C}\Gamma_n} H_c$

whose image under the quotient map corresponds to m in the isomorphism of Proposition 4.20. Now $M_1 := M^{\Gamma_{n-1}}$ is isomorphic to R^n as a $\mathbb{C}\Gamma$-module and we get

$$\phi^{\sharp}(F)(m) = (e_{\text{diag}} \otimes e_{\Gamma} \circ \text{sym}(f)(x_1, y_1) \circ e_{\text{diag}} \otimes e_{\Gamma})M_1. \tag{4.25}$$

The subspace $\mathbb{C}_{\text{diag}} \otimes \mathbb{C}e$ in R^n corresponds to the subspace $e'M \subset M$ and so (4.25) is equal to

$$e'\text{sym}(f)(x_1, y_1)e'M$$

and since $e'M$ is invariant under the action of S_n this equals

$$\frac{1}{n}e'\sigma(f)e'M = \frac{1}{n}\sigma^{\sharp}(f)e'M = \frac{1}{n}\sigma^{\sharp}(f)(m).$$

We have established that $\phi^{\sharp}(F) = \frac{1}{n}\sigma^{\sharp}(f)$ for all $f \in \mathbb{C}[x, y]^{\Gamma}$ and so $\phi^{\sharp}$ is surjective. □

Bringing together Theorem 4.10 and Theorem 4.23 we get an immediate corollary.

Corollary 4.25. *There is an isomorphism between X_c and $\mathcal{N}(\lambda'(c), \epsilon_{\infty} + n\delta)$ which is Poisson up to nonzero scalar multiple and in particular identifies symplectic leaves.*

Thus by Lemma 4.6 we can describe the symplectic leaves of X_c in terms of the representation type strata of $\mathcal{N}(\lambda'(c), \epsilon_{\infty} + n\delta)$, and so using results from Section 2.3 we can obtain basic numerical information about the symplectic leaves.

4.4 Remarks

1. The shifting trick used in Theorem 4.2 is standard in symplectic geometry. The reason we cannot use this trick for more general orbits, that is, closed orbits which may not be fixed points, is that the local normal form may not be preserved. More precisely, although the local normal form exists for M it may not exist for $M \times -\mathcal{L}$. Of course it is trivial that if $\mathcal{L}$ is just a point then the local normal form will exist for the product. In general one might require some extra structure on $\mathcal{L}$ cf. Section 2.4. Certainly there are cases when $\mathcal{L}$ will have a hyper-Kähler structure, see [50], but even this will not guarantee that $M \times -\mathcal{L}$ has a local normal form. This is because the proof of Corollary 2.36 uses the fact that we are working in a vector space, cf. Proposition 2.35.

2. The idea of Proposition 4.13 to identify a Marsden-Weinstein reduction over a closed coadjoint orbit to one defined over and single point is part of a more general phenomenon. It is pointed out in [18, Remark 9.2] that, in the notation of Example 2.25, if one took any closed coadjoint orbit $\mathcal{L}$ then $\mu_\alpha^{-1}(\mathcal{L})//\,\mathrm{G}(n\delta)$ is isomorphic to a reduction $\mathcal{N}(\lambda', \alpha')$ associated to some quiver Q', fixed point λ' and dimension vector α'.

Chapter 5

Examples

We calculate some examples using Corollary 4.25, working out for which values of c the variety X_c is smooth. By Theorems 1.44 and 1.43, X_c is smooth if and only if it is symplectic and so there will only be nontrivial leaves when X_c is singular. In these cases we work out the number and dimension of the symplectic leaves. We use this basic numerical data to show that the map Ω from (3.1) is injective in certain cases.

The simplest case of a wreath product, Γ_n, to consider is when the group Γ is trivial, that is, where the there is only the symmetric group S_n acting by its permutation representation on $(\mathbb{C}^2)^n$. By Proposition 1.25 and (1.6), X_c is smooth for all nonzero values of c. We therefore move onto the next simplest case.

5.1 The symplectic leaves of X_c for $S_n \wr \mathbb{Z}_2$

Let $\Gamma_n = S_n \ltimes (\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2)$ where we identify $\mathbb{Z}_2$ with $< \gamma = -\mathrm{Id}_{\mathbb{C}^2} >$. There are two conjugacy classes of symplectic reflections, see Notation 1.22, and we write $c = (c_1, c_\gamma)$ (in the notation of Notation 1.22, we have set $c_\gamma = \underline{c}$). Recall the notation from the beginning of Section 4.2. Since Γ is abelian and has order 2, the vector δ equals $(1,1)$.

Corollary 4.25 gives us $X_c \cong \mathcal{N}(\lambda'(c), \epsilon_\infty + n\delta)$. The latter is the Marsden-Weinstein reduction associated to the quiver

$$Q:\ \infty \longrightarrow 0 \rightleftarrows 1$$

with dimension vector $\epsilon_\infty + n\delta = (1, n, n)$ and at parameter $\lambda := \lambda'(c) = (nc_1, -c_1 + c_\gamma, -c_\gamma)$.

We refer the reader to the definitions and results laid out in Section 2.3. We have seen that $X_0 = (\mathbb{C}^2)^{\oplus n}/\Gamma_n$ is singular (see the paragraph following Lemma 1.5) and its symplectic leaves are known, Example 1.37.

Theorem 5.1. *Let X_c be as above and assume that $c \neq (0,0)$. Then*

1. *X_c is singular if and only if $c_1 = 0$ or $c_\gamma = \pm m c_1$ for some integer, m, such that $0 \leq m \leq n-1$.*

2. *If $c_1 = 0$ then the symplectic leaves of X_c are parametrised by P_n, the set of partitions of n. For each $\sigma \in P_n$, the corresponding leaf, $\mathcal{S}_\sigma$ has dimension $2l(\sigma)$, where $l(\sigma)$ is the length of σ.*

3. *If $c_\gamma = \pm m c_1$ for some $0 \leq m \leq n-1$ then the leaves are parametrised by the set $S = \{k \in \mathbb{Z}_{\geq 0} : km + k^2 \leq n\}$. For $k \in S$ the corresponding leaf, $\mathcal{S}_k$ has dimension $2(n - km - k^2)$.*

Remark 5.2. The behaviour of X_c when $c_1 = 0$ is always analogous to that occurring in the above theorem. Denote by $H_{\underline{c}}$ the symplectic reflection algebra associated to the pair (L, ω_L) acted on by the group Γ, with centre $Z_{\underline{c}}$. When $c_1 = 0$ one can calculate that H_c, the algebra defined on the wreath product, Γ_n, is isomorphic to $\left(\bigotimes_n H_{\underline{c}}\right) * S_n$ - such a result has been proved in the $t = 1$ case, [28]. Therefore one can see that $X_c \cong \operatorname{Sym}^n \operatorname{Max} Z_{\underline{c}}$, and one would expect that the symplectic leaves of X_c occur as orbit type strata of $(S_n \cdot (\mathcal{S}_1 \times \cdots \times \mathcal{S}_n))/S_n$ where the $\mathcal{S}_i$ are symplectic leaves of the ith copy of $\operatorname{Max} Z_{\underline{c}}$, cf. Example 1.37. In the case where $|\Gamma| = 2$, the varieties $\operatorname{Max} Z_{\underline{c}}$ are smooth for nonzero values of $\underline{c}$, using arguments identical to those of Proposition 1.25. Therefore each $\operatorname{Max} Z_{\underline{c}}$ is its own symplectic leaf and the leaves of X_c should simply correspond to partitions of n, which is the case by Theorem 5.1 (ii) above.

The remainder of this section will be devoted to proving Theorem 5.1. We fix notation: let $\alpha = (1, n, n)$ and let $\epsilon_\infty = (1,0,0), \epsilon_0 = (0,1,0)$ and $\epsilon_1 = (0,0,1)$. Let $I = \{\infty, 0, 1\}$. Then one sees easily that $(\epsilon_i, \epsilon_i) = 2$ for all $i \in I$ and also that $(\epsilon_\infty, \epsilon_0) = -1$, $(\epsilon_\infty, \epsilon_1) = 0$ and $(\epsilon_0, \epsilon_1) = -2$.

We note some identities which will appear several times in our calculations. Let $\beta, \theta \in \mathbb{N}^I$. Then

$$(\beta, \theta) = 2(\beta_\infty \theta_\infty + \beta_0 \theta_0 + \beta_1 \theta_1) - (\beta_\infty \theta_0 + \beta_0 \theta_\infty) - 2(\beta_0 \theta_1 + \theta_0 \beta_1)$$

and

$$p(\beta) = 1 - \frac{1}{2}(\beta, \beta) = 1 - (\beta_\infty^2 + \beta_0^2 + \beta_1^2) + \beta_\infty \beta_0 + 2\beta_0 \beta_1.$$

Let s_i be the reflection at vertex i. Then we have

$$\begin{aligned} p(s_i\beta) &= p(\beta - (\beta,\epsilon_i)\epsilon_i) = 1 - \frac{1}{2}(\beta - (\beta,\epsilon_i)\epsilon_i, \beta - (\beta,\epsilon_i)\epsilon_i) \\ &= 1 - \frac{1}{2}((\beta,\beta) - 2(\beta,\epsilon_i)^2 + (\beta,\epsilon_i)^2(\epsilon_i,\epsilon_i)) = p(\beta). \end{aligned} \tag{5.1}$$

We begin with a preliminary lemma bringing together some easy facts about roots for the quiver Q. Recall that *Re*, *Im* and $\mathcal{F}$ denote the real roots, imaginary roots and the fundamental region respectively.

Lemma 5.3.

(1) The set $\{\beta \in Re^+ : \beta \leq \alpha, \beta_\infty = 0\}$ *equals* $\{(0,m,m+1) : 0 \leq m \leq n-1\} \cup \{(0,m+1,m) : 0 \leq m \leq n-1\}$*;*

(2) If $\beta \in \mathcal{F}$ *then* $\beta_i \geq 0$ *for all* $i \in I$*;*

(3) If $\beta \in Im^+$ *then* $p(\beta) \geq 1$*;*

(4) The set $\{\beta \in Im^+ : \beta_\infty = 0\}$ *equals* $\{(0,m,m) : m \geq 1\}$*, and* $p(0,m,m) = 1$ *for all* $m \geq 1$*;*

(5) The set $\{(1,m,m) : m \geq 1\} \subset Im^+$ *and* $p(1,m,m) = m$ *for all* $m \geq 1$*;*

(6) The set $\{(1,m,m+1) : m \geq 2\} \cup \{(1,m+1,m) : m \geq 1\} \subset Im^+$*,* $p(1,m,m+1) = m-1$ *and* $p(1,m+1,m) = m$*. The element* $(1,1,2) \in Re^+$*.*

Proof.

(1) By (5.1) $\beta \in Re$ implies that $p(\beta) = 0$. For any $k,l \in \mathbb{N}$, $p(0,k,l) = 1 - \frac{1}{2}(2k^2 - 4kl + 2l^2) = 1 - (k-l)^2$ which is zero if and only if $|k-l| = 1$. To see that all such values of k and l do in fact yield real roots we simply note that for $m \geq 1$

$$(0,m,m+1) = \begin{cases} (s_1 s_0)^{\frac{m}{2}}\epsilon_1 & \text{if m is even} \\ s_1(s_0 s_1)^{\frac{m-1}{2}}\epsilon_0 & \text{if m is odd} \end{cases}$$

$$\text{and } (0,m+1,m) = \begin{cases} (s_0 s_1)^{\frac{m}{2}}\epsilon_0 & \text{if m is even} \\ s_0(s_1 s_0)^{\frac{m-1}{2}}\epsilon_1 & \text{if m is odd.} \end{cases}$$

(2) Let $\beta \in \mathcal{F}$ so that $(\beta, \epsilon_i) \leq 0$ for each $i \in I$. This is equivalent to the following three conditions:

$$2\beta_\infty \leq \beta_0 \tag{a}$$

$$2\beta_0 \leq \beta_\infty + 2\beta_1 \tag{b}$$

$$\beta_1 \leq \beta_0. \tag{c}$$

Now (b) implies that $\beta_\infty \geq 2(\beta_0 - \beta_1)$, which is greater than zero by (c). From (a) we get $\beta_0 \geq 2\beta_\infty \geq 0$, and finally combining (a) and (b) yields $\beta_1 \geq \beta_0 - \frac{1}{2}\beta_\infty \geq \frac{3}{2}\beta_\infty \geq 0$.

(3) Suppose first that $\beta \in \mathcal{F}$. Since β is positive by (2) we obtain from equations $(a) - (c)$ above:

$$\begin{aligned} 2\beta_\infty^2 &\leq \beta_\infty\beta_0 \\ 2\beta_0^2 &\leq \beta_\infty\beta_0 + 2\beta_0\beta_1 \\ 2\beta_1^2 &\leq 2\beta_0\beta_1. \end{aligned}$$

These together imply that

$$\begin{aligned} p(\beta) &= 1 - \frac{1}{2}\left(2\beta_\infty^2 + 2\beta_0^2 + 2\beta_1^2 - 2\beta_\infty\beta_0 - 4\beta_0\beta_1\right) \\ &\geq 1 - 0. \end{aligned}$$

If we suppose that $\beta \in -\mathcal{F}$ then β is negative and one sees easily that $p(\beta) = p(-\beta) \geq 1$. Since every imaginary root is obtained from $\mathcal{F} \cup -\mathcal{F}$ by sequence of reflections it follows from (5.1) that $p(\beta) \geq 1$ for every $\beta \in Im$.

(4) This is a straightforward calculation using (3). Let $\beta \in Im^+$ and suppose that $\beta_\infty = 0$. Then

$$\begin{aligned} p(\beta) &= 1 - \frac{1}{2}\left(2\beta_0^2 + 2\beta_1^2 - 4\beta_0\beta_1\right) \\ &= 1 - (\beta_0 - \beta_1)^2. \end{aligned}$$

Therefore $p(\beta) \geq 1$ if and only if $\beta_0 = \beta_1$ and one can check that $\{(0, m, m) : m \geq 1\} \subset \mathcal{F}$ and that $p(0, m, m) = 1$.

(5) The element $(1, 1, 1) = s_\infty(0, 1, 1)$ so is contained in Im^+ and $p(1, 1, 1) = 1$. For $m \geq 2$, $(1, m, m) \in \mathcal{F}$ and $p(1, m, m) = m$ can be verified by calculation.

(6) We have $(1,1,2) = s_1 s_\infty \epsilon_0$ and $(1,m,m+1) = s_1 s_0(1,m-1,m-1)$ for $m \geq 2$. Also, $(1,m+1,m) = s_0(1,m,m)$ for $m \geq 1$. Therefore this result follows easily from (5).

□

Smoothness

Let $R^i_\lambda = \{\beta \in R^+ : \beta \leq \alpha,\ \lambda \cdot \beta = 0 \text{ and } \beta_\infty = i\}$. Recall that Σ_λ denotes the roots of Q which are simple dimension vectors for Π_λ. We have the following easy lemma.

Lemma 5.4. *Suppose that $\lambda \neq 0$, then*

$$R^0_\lambda = \begin{cases} \{(0,k,k) : 1 \leq k \leq n\} & \textit{if } c_1 = 0 \\ \{(0,m,m+1)\} & \textit{if } c_1 \neq 0 \textit{ and } c_\gamma = -mc_1 \textit{ for } 0 \leq m \leq n-1 \\ \{(0,m,m-1)\} & \textit{if } c_1 \neq 0 \textit{ and } c_\gamma = mc_1 \textit{ for } 1 \leq m \leq n-1 \\ \emptyset & \textit{otherwise.} \end{cases}$$

If $c_1 = 0$ then $R^0_\lambda \cap \Sigma_\lambda = \{(0,1,1)\}$, and when $c_1 \neq 0$ we have $R^0_\lambda \subseteq \Sigma_\lambda$.

Proof. The first part is a straightforward consequence of Lemma 5.3 (1) and (4). The final statement is implied by Theorem 2.29 and the fact that $p(0,k,k) = 1$, Lemma 5.3 (4). □

Corollary 5.5. *Suppose that $\lambda \neq 0$ and that $\rho \leq \alpha$ is a positive vector. Suppose we have an expression $\rho = \rho' + \beta_1 + \cdots + \beta_t$ where $\rho' \in R^1_\lambda$ and $\beta_i \in R^0_\lambda$ for each i. If $c_1 \neq 0$ then $\beta_1 = \cdots = \beta_t$ and if $c_1 = 0$ then the β_i are equal to $(0,k,k)$ for some k. If we assume that $c_1 = 0$ and the $\beta_i \in \Sigma_\lambda$ then the β_i are all equal to $(0,1,1)$.*

We now are in a position to describe for which values of c the variety X_c is singular.

Theorem 5.6. *X_c (with $c \neq 0$) is singular if and only if $c_1 = 0$ or $c_\gamma = \pm mc_1$ for some $0 \leq m \leq n-1$.*

Proof. By Lemma 2.31 (2), there are only two possibilities for the unique decomposition of α from Theorem 2.30 (i), either

$$\alpha = \alpha \tag{5.2}$$

or

$$\alpha = \epsilon_\infty + (0,1,1) + \cdots + (0,1,1). \tag{5.3}$$

Suppose that $\mathcal{N}(\lambda, \alpha)$ is singular. If the unique decomposition of α is of the form (5.2) then by Corollary 4.8 there must exist a decomposition $\alpha = \alpha' + \beta_1 + \cdots + \beta_t$ where $\alpha', \beta_i \in R_\lambda^+$. Without loss of generality we can assume that $\alpha' \in R_\lambda^1$ and $\beta_i \in R_\lambda^0$ for each i. Therefore by Corollary 5.5 and Lemma 5.4, c must take one of the forms listed above.

On the other hand, if (5.3) is the unique decomposition of α then, in particular, $(0, 1, 1) \cdot \lambda = 0$ so that $c_1 = 0$.

We now prove the reverse implication. Suppose that $c_1 = 0$, then (5.3) is a decomposition of α and an easy verification using Theorem 2.29 shows that both ϵ_∞ and $(0, 1, 1)$ are in Σ_λ. Therefore, since $p(0, 1, 1) = 1$, $\mathcal{N}(\lambda, \alpha)$ is singular by Corollary 4.8.

Now suppose that $c_\gamma = \pm mc_1$ for some $0 \leq m \leq n-1$. Then (5.2) is the unique decomposition for α and in particular $\alpha \in \Sigma_\lambda$. Then by Lemma 5.3 (1) there is a unique real root $\beta \leq \alpha$ with $\beta_\infty = 0$ such that $\lambda \cdot \beta = 0$. By Lemma 5.4 $\beta \in \Sigma_\lambda$. By Lemma 5.3 (5) and (6), $\alpha - \beta$ is a positive imaginary root, and clearly $\lambda \cdot (\alpha - \beta) = 0$. Choose t to be the largest natural number so that $\alpha - t\beta$ is a positive root, and call this root ρ. Then we claim that ρ is a simple dimension vector. If this were not true then by Theorem 2.29 we could write ρ as a sum of smaller roots, $\rho = \rho' + \sigma_1 + \cdots + \sigma_k$, where $\rho' \in R_\lambda^1$ and $\sigma_i \in R_\lambda^0$. By Corollary 5.5 we have $\sigma_1 = \cdots = \sigma_k$, and we call this root σ. However, $\lambda \cdot \sigma = 0$ implies that $\sigma = \beta$, Lemma 5.4. Therefore by the maximality of t, ρ is a simple dimension vector and $\alpha = \rho + t\beta$ is a nontrivial decomposition into a sum of simple dimension vectors. Corollary 4.8 then implies that $\mathcal{N}(\lambda, \alpha)$ is singular. $\square$

Symplectic Leaves

We calculate some basic facts about the symplectic leaves of $X_c \cong \mathcal{N}(\lambda, \alpha)$; by Lemma 4.6 the symplectic leaves of $\mathcal{N}(\lambda, \alpha)$ are the representation type strata. We can work out the representation type strata using roots as explained at the end of Section 4.1.

We retain the assumptions from above and in particular we shall assume that $c \neq 0$ (which implies that $\lambda \neq 0$). We define $\Sigma_\lambda^i = \{\beta \in \Sigma_\lambda : \beta \leq \alpha \text{ and } \beta_\infty = i\}$, by Lemma 5.4 $\Sigma_\lambda^0 = R_\lambda^0$.

Proposition 5.7.

(1) If $c_1 = 0$ the symplectic leaves of X_c are parametrised by P, the set of partitions of n, and for each $\sigma \in P$ the corresponding leaf, $\mathcal{R}_\sigma$, has dimension $2l(\sigma)$.

(2) If $c_1 \neq 0, c_\gamma = \pm mc_1$ for some $0 \leq m \leq n-1$ then the symplectic leaves of X_c are parametrised by the set $S = \{\alpha - k\beta_\lambda : k \geq 0, km + k^2 \leq n\}$. For each $s = \alpha - k\beta_\lambda \in S$ the symplectic leaf, $\mathcal{R}_s$, corresponding to s has dimension $2(n - km - k^2)$.

Proof.

(1) If $c_1 = 0$ then, as noted in the proof of Theorem 5.6, $\alpha = \epsilon_\infty + (0,1,1) + \cdots + (0,1,1)$ is a sum of elements of Σ_λ. We claim that there are no other decompositions of α as a sum of simple dimension vectors. By Lemma 2.31 and Theorem 2.30 (2) any other decomposition is a refinement of this one, and in particular this would imply that either ϵ_0 or ϵ_1 are in Σ_λ. However, since $\lambda = (0, c_\gamma, -c_\gamma)$ and $\lambda \neq 0$ this cannot happen, proving the claim. Since $p(0,1,1) = 1$ there are infinitely many nonisomorphic simple Π_λ-modules with dimension vector $(0,1,1)$, Lemma 2.27. Thus representation types correspond to partitions of n and the dimension of the stratum associated to each partition follows from Proposition 2.28.

(2) Let $c_1 \neq 0$ and $c_\gamma = \pm mc_1$ for some $0 \leq m \leq n-1$ and let β_λ be the corresponding element of $R^0_\lambda = \Sigma^0_\lambda$ (Lemma 5.4). As noted in the proof of Theorem 5.6, $\alpha \in \Sigma_\lambda$. By Corollary 5.5 every decomposition of α into a sum of simple dimension vectors is of the form $\alpha' + t\beta_\lambda$ for some $\alpha' \in \Sigma^1$ and some $t \geq 0$. For any $k \geq 0$ a simple calculation shows that $p(\alpha - k\beta_\lambda) = n - km - k^2$. Let $S = \{\alpha - k\beta_\lambda : k \geq 0, km + k^2 \leq n\}$. We note that for any $s, s' \in S$, $p(s) \geq p(s')$ implies $s \geq s'$. The set S consists of positive vectors such that $s \cdot \lambda = 0$ for all $s \in S$. In fact S consists of positive roots since, for all k such that $km + k^2 \leq n$,

$$\alpha - k\beta_\lambda = \begin{cases} (s_1 s_0)^k (1, n - km - k^2, n - km - k^2) & \text{if } c_\gamma = mc_1 \text{ for } 1 \leq m \leq n-1 \\ s_0 (s_1 s_0)^{k-1} (1, n - km - k^2, n - km - k^2) & \text{if } c_\gamma = -mc_1 \text{ for } 1 \leq m \leq n-1, \end{cases}$$

and $(1, n - km - m^2, n - km - m^2)$ is an imaginary root by Lemma 5.3 (5).

We show that $S \subseteq \Sigma_\lambda$. To see this let $s \in S \backslash \Sigma_\lambda$. Then by Theorem 2.29 and Lemma 5.4 there exists a decomposition of s into form $s' + l\beta_\lambda$ for some $l \geq 1$ such that $p(s') + lp(\beta_\lambda) \geq p(s)$. However, we know that β_λ is a real root by Lemma 5.3 (1) so that $p(\beta_\lambda) = 0$. Therefore $s' \leq s$ and $p(s') \geq p(s) \geq 0$ and so $s' \in S$. Now since $s, s' \in S$ we have that $p(s') \geq p(s)$ implies $s' \geq s$. Thus $s = s'$, a contradiction. Thus for any $s \in S$, $\alpha = s + (\alpha - s)$ is a decomposition of α as a sum of elements from Σ_λ.

Let $\alpha = \alpha' + k\beta_\lambda$ be a decomposition with $\alpha' \in \Sigma_\lambda$. In particular α' is a positive root so $p(\alpha') \geq 0$ by Lemma 5.3 (3). Therefore $\alpha' \in S$. We conclude that the complete set of decompositions of α into a sum of simple dimension vectors is $\{(\alpha - k\beta_\lambda) + k\beta_\lambda : km + k^2 \leq n\}$. Moreover, each such decomposition yields a single representation type because $p(\beta_\lambda) = 0$ (see Lemma 2.27). Therefore the leaves are in one-to-one correspondence with elements of S and for each $s = \alpha - k\beta_\lambda \in S$ its corresponding leaf, $\mathcal{R}_s$, has dimension $2p(s) = 2(n - km - k^2)$ by Proposition 2.28.

□

We can apply this theorem to associated varieties as discussed in Chapter 3. Recall that we have a map

$$\Omega : \{\text{Symplectic leaves in } X_c\} \dashrightarrow \{\text{Conjugacy classes of subgroups in } G\}$$

from (3.1) which is induced from the map

$$\{\text{Poisson prime ideals of } Z_c\} \rightarrow \{\text{Poisson prime ideals of } SV^G\};\ P \mapsto \sqrt{\operatorname{gr} P}. \tag{5.4}$$

Corollary 5.8. *Let X_c be the variety associated to $\Gamma_n = S_n \ltimes (\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2)$ acting on $(\mathbb{C}^2)^{\oplus n}$. Suppose that $c_1 \neq 0$. Then Ω is injective.*

Proof. We begin by noting that if X_c is smooth then it contains one symplectic leaf so the result is trivially true. Suppose that $c_1 \neq 0$ and X_c is singular so that $c_\gamma = mc_1$ or $-mc_1$ for some $0 \leq m \leq n-1$. Let $\mathcal{R}, \mathcal{S}$ be symplectic leaves in X_c. By the theorem their dimensions are equal to $2(n - km - k^2)$ and $2(n - lm - l^2)$ respectively for some $k, l \in \mathbb{Z}_{\geq 0}$. The expressions for the dimensions of the leaves imply that $\operatorname{Dim} \mathcal{R} = \operatorname{Dim} \mathcal{S}$ if and only if $k = l$ and this happens if and only if $\mathcal{R} = \mathcal{S}$ by the theorem. Thus the symplectic leaves of X_c have distinct dimensions. It follows from Theorem 1.43 (ii) that the varieties defined by the Poisson prime ideals of Z_c have distinct dimensions. Let P be a Poisson prime ideal of Z_c. By [49, Proposition 6.6] we have $\operatorname{Dim} \mathcal{V}(P) = \operatorname{Dim} \mathcal{V}(\sqrt{\operatorname{gr} P})$. In particular the map (5.4) is injective and so Ω is injective also. □

5.2 Comparison between symplectic leaves and representations in the $S_2 \wr \mathbb{Z}_2$ case

We calculate some irreducible representations of H_c in the case of $S_2 \wr \mathbb{Z}_2$ and compare these with the symplectic leaves of X_c. We emphasise that the calculations in this section do not give all possible irreducible representations for H_c but are intended as an indication of the nature of small representations occurring.

Let us begin with some notation. Let $s = s_{12} \in S_2$ and we write $\mathbb{Z}_2 = \{\pm 1\}$ (that is, the group operation is multiplicative). The wreath product $S_2 \wr \mathbb{Z}_2$ is isomorphic to the dihedral group of order eight, D_8. We can give a presentation of D_8 as $\langle a, b : a^4 = b^2 = 1, bab = a^{-1}\rangle$ and then the isomorphism is given by

$$S_2 \wr \mathbb{Z}_2 \to D_8; s \cdot (-1, 1) \mapsto a, s \mapsto b. \tag{5.5}$$

Under this isomorphism the conjugacy classes of symplectic reflections for the action of $S_2 \wr \mathbb{Z}_2$ on $\mathbb{C}^2 \oplus \mathbb{C}^2$ are $\{b, ba^2\}$ and $\{ba, ba^3\}$. The irreducible representations of D_8 are well known, we list them below. There are four one-dimensional ones:

$$\tau : \begin{array}{c} a \mapsto (1) \\ b \mapsto (1) \end{array}, \quad -\tau : \begin{array}{c} a \mapsto (-1) \\ b \mapsto (-1) \end{array}, \quad \sigma : \begin{array}{c} a \mapsto (1) \\ b \mapsto (-1) \end{array}, \quad -\sigma : \begin{array}{c} a \mapsto (-1) \\ b \mapsto (1) \end{array},$$

and one two-dimensional one:

$$\rho : a \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The symplectic reflection algebra, H_c, corresponding to $S_2 \wr \mathbb{Z}_2$ has generators x_1, y_1, x_2, y_2, a, b which satisfy the relations

$$a^4 = b^2 = 1, \quad bab = a^{-1}, \tag{5.6}$$

$$\begin{gathered} ax_1a^{-1} = -x_2, \quad ax_2a^{-1} = x_1, \quad ay_1a^{-1} = -y_2, \quad ay_2a^{-1} = y_1 \\ bx_1b = x_2, \quad bx_2b = x_1, \quad by_1b = y_2, \quad by_2b = y_1, \end{gathered} \tag{5.7}$$

$$[x_1, x_2] = [y_1, y_2] = 0, \quad [x_1, y_2] = -\frac{1}{2}c_1 b + \frac{1}{2}c_1 ba^2, \quad [x_1, y_1] = \frac{1}{2}c_1 b + \frac{1}{2}c_1 ba^2 + c_\gamma ba. \tag{5.8}$$

The relations for $[x_2, y_1]$ and $[x_2, y_2]$ are derived from (5.8) by using the fact that conjugation by b in H_c swaps x_1 and x_2 and swaps y_1 and y_2. It is straightforward to deduce from (5.7) the following relations hold in H_c for $i = 1, 2$

$$a^2 x_i a^2 = -x_i, \quad a^2 y_i a^2 = -y_i. \tag{5.9}$$

We look for small representations of H_c (in the sense described in Section 1.3) and compare these with the information we have about the symplectic leaves of X_c from Proposition 5.7. The next result is simply Theorem 5.6 and Proposition 5.7 calculated for $S_2 \wr \mathbb{Z}_2$.

Proposition 5.9. *For $c \neq 0$ the variety X_c is singular precisely when*

$$(1)\ c_1 = 0,\ (2)\ c_\gamma = 0,\ (3)\ c_1 + c_\gamma = 0 \text{ or } (4)\ c_1 - c_\gamma = 0.$$

In all cases there are two symplectic leaves, one of which is the smooth locus and has dimension 4. In the first two cases the remaining leaf has dimension 2, and in the last two it has dimension 0.

We are of course excluding the very important case of $c = 0$. It is easy to calculate using Theorem 4.25 the number and dimensions of leaves. Furthermore, in [11, 7.6] Brown and Gordon give an explicit description of the factor algebras $SV * G/\mathfrak{m}SV * G$ where $\mathfrak{m} \in \operatorname{Max} SV^G$.

We attempt to find representations of H_c by first taking representations for $S_2 \wr \mathbb{Z}_2 = D_8$ and determining whether this extends to a representation of H_c. We begin with a couple of remarks. Let $\theta : \mathbb{C}D_8 \to \operatorname{End}_{\mathbb{C}}(M)$ be a representation of D_8. Now (5.8) implies that

$$\operatorname{tr}_M \theta(-c_1 b + c_1 ba^2) = \operatorname{tr}_M \theta(\frac{1}{2}c_1 b + \frac{1}{2}c_1 ba^2 + c_\gamma ba) = 0. \tag{5.10}$$

This can be strengthened when $\theta(a^2)$ is a central endomorphism of M. Then (5.9) implies that any extension of θ to H_c must satisfy $x_i|_M = y_j|_M = 0$ for all i, j which in turn implies

$$\theta(-c_1 b + c_1 ba^2) = \theta(\frac{1}{2}c_1 b + \frac{1}{2}c_1 ba^2 + c_\gamma ba) = 0 \tag{5.11}$$

by (5.8).

We deal with each case in turn.

$\mathbf{c_1 = 0}$: Here there is great deal of symmetry so the picture is quite complete. The relations (5.8) yield $[x_1, y_2] = [x_2, y_1] = 0$ so that $H_c \cong (H_{c_\gamma} \otimes H_{c_\gamma}) * S_2$ where H_{c_γ} is the symplectic reflection algebra for the action of $\mathbb{Z}_2$ on $\mathbb{C}^2$ at parameter c_γ. Since $c_\gamma \neq 0$ all irreducible representations of H_{c_γ} are isomorphic to $\mathbb{C}\mathbb{Z}_2$ - this follows from Theorem 1.17 by mimicking the argument used in Proposition 1.25. Let V be an irreducible representation of H_{c_γ}, and let T, S be the irreducible representations of S_2. There is an S_2-action on $V \otimes V$ by swapping tensorands. Let $M_1(V) = V \otimes V \otimes T$ and $M_2(V) = V \otimes V \otimes S$. Let $p, q \in H_{c_\gamma}$ and $g \in S_2$, then $M_1(V)$ becomes an H_c-module via

$$(p \otimes q \otimes g) \cdot (v \otimes w \otimes t) = (p \otimes q) \cdot (g(v \otimes w)) \otimes g \cdot t$$

for all $v, w \in V$ and $t \in T$. Similarly, $M_2(V)$ is an H_c-module. The irreducibility of V implies that both $M_1(V)$ and $M_2(V)$ are irreducible H_c-modules. If we consider both of these as modules over $D_8 \subset H_c$ then we get

$$M_1(V) = \tau \oplus -\sigma \oplus \rho, \quad M_2(V) = -\tau \oplus \sigma \oplus \rho$$

so that as a $\mathbb{C}D_8$-module $M_1(V) \oplus M_2(V)$ is the regular representation. The irreducible representations of H_{c_γ} are parametrised by X_{c_γ} which has dimension two. One might expect that the pairs $\{M_1(V), M_2(V)\}$ should correspond to points in the two-dimensional leaf of X_c. There are potentially many more small irreducible representations of H_c, and we know no way of establishing whether or not this is the case. However, by using the identities (5.10) and (5.11) one can verify, using case-by-case analysis, that H_c has no representations of dimension less than or equal to three.

$\mathbf{c}_\gamma = \mathbf{0}$: There seems to be no obvious way to use the structure of H_c in order to find small representations in this case. However, using (5.10) and (5.11) one can calculate that H_c has no representations of dimension less than or equal to three, again by brute force. There do exist four-dimensional representations. For example

$$M_1 : a \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; b \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$x_1 \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \; y_1 \mapsto \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2}c_1 \\ 0 & 0 & 0 & \frac{1}{2}c_1 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}c_1 & -\frac{1}{2}c_1 & 0 & 0 \end{pmatrix};$$

and

$$M_2 : a \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; b \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$x_1 \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; y_1 \mapsto \begin{pmatrix} 0 & 0 & \frac{1}{2}c_1 & 0 \\ 0 & 0 & -\frac{1}{2}c_1 & 0 \\ -\frac{1}{2}c_1 & \frac{1}{2}c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since there are no representations of smaller dimension both M_1 and M_2 are irreducible and we note that $M_1 \oplus M_2 \cong \mathbb{C}D_8$ as modules for D_8. Since there is a two-dimensional leaf in this case there will be infinitely many isomorphism classes of small irreducible modules. Thus we should not draw too many conclusions from these examples: there could be other $\mathbb{C}D_8$-modules on which one can introduce a H_c-module structure.

$\mathbf{c_1 + c_\gamma = 0}$: There are two irreducible H_c-modules which are easy to discover, namely take the $\mathbb{C}D_8$-modules τ and σ and extend to an action of H_c by allowing the $x_i s$ and $y_j s$ to act by zero. The only nontrivial relations to check are those of (5.8). Now there is a six-dimensional representation:

$$M: \; a \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \; b \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$x_1 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \; y_1 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 & 0 & -k \\ 0 & 0 & 0 & 0 & -k & 0 \\ 0 & k & 0 & k & 0 & 0 \\ -k & 0 & -k & 0 & 0 & 0 \end{pmatrix},$$

where $k = \frac{1}{2}c_1$. As a D_8-module M is isomorphic to $-\tau \oplus -\sigma \oplus \rho \oplus \rho$. The fact that M is irreducible follows from (5.10) and (5.11) - these equations imply that H_c has no two-dimensional or three-dimensional modules. Thus if we consider the Jordan-Hölder series of M we get a contradiction unless M is simple. The direct sum of the two one dimensional representations and M yields the regular representation. Here the singular locus of X_c is zero-dimensional so there are only finitely many irreducible small representations.

$\mathbf{c_1 - c_\gamma = 0}$: There are two one-dimensional modules obtained by taking $-\tau$ and $-\sigma$ and extending these to H_c-modules by allowing the $x_i s$ and $y_j s$ act by zero. There is a six-dimensional representation:

$$M: \; a \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; b \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$x_1 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \; y_1 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -k \\ 0 & 0 & 0 & 0 & -k & 0 \\ 0 & 0 & 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 & k & 0 \\ 0 & -k & 0 & -k & 0 & 0 \\ k & 0 & k & 0 & 0 & 0 \end{pmatrix},$$

where $k = \frac{1}{2}c_1$. One sees that M is equal to $\tau \oplus \sigma \oplus \rho \oplus \rho$ as a D_8-module. As in the previous example the fact that M is irreducible follows from (5.10) and (5.11) which imply that H_c has no two-dimensional or three-dimensional modules. Thus we have that the direct sum of these three irreducible modules is isomorphic to the regular representation. Here also the the singular locus of X_c is zero-dimensional so there are finitely many small irreducible representations of H_c.

Remark 5.10. It follows from [33, Theorem 4.8(1)] that when G is a wreath product then for any $\mathfrak{m} \in X_c$ there exist H_c-modules $M_1, \dots, M_t$ which are annihilated by $\mathfrak{m}$ such that $M_1 \oplus \cdots \oplus M_t$ is isomorphic to the regular representation. This explains the phenomena which occur in the examples above.

5.3 Questions

Let (V, ω, G) be an indecomposable symplectic triple. Let H be a symplctic reflection algebra associated to (V, ω, G). Denote the centre of H by Z. Recall the map χ from (1.5). For any maximal ideal $\mathfrak{m} \subset Z$ the fibre $\chi^{-1}(\mathfrak{m})$ equals the set of irreducible H-modules annihilated by $\mathfrak{m}$.

We pose the following as a generalisation of Theorem 1.17. Certainly the example of $c_1 = 0$ above offers some evidence that this might be true, but as noted in Remark 5.2 this is a somewhat special case.

Question 1. *Suppose that $\mathfrak{m}, \mathfrak{n} \in \operatorname{Max} Z$ belong to the same symplectic leaf. Are $\chi^{-1}(\mathfrak{m})$ and $\chi^{-1}(\mathfrak{n})$ equal as sets of G-modules?*

Finally we consider the map Ω from (3.1). As we have shown in Corollary 5.8 this map is often injective when G is a Weyl group of type B_n and $V = \mathfrak{h} \oplus \mathfrak{h}^*$ where $\mathfrak{h}$ is Cartan subalgebra associated to G. When G is the Weyl group of type A_n then Ω is always injective by Proposition 1.25. On the other hand we have noted in Remarks 3.5 that Ω fails to be injective in some relatively basic examples where G is not a Weyl group.

Question 2. *Let G be a Weyl group acting on the double $V = \mathfrak{h} \oplus \mathfrak{h}^*$. Is the map Ω injective for all values of c?*

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